NBER WORKING PAPER SERIES

BANK RUNS, FRAGILITY, AND REGULATION

Manuel Amador
Javier Bianchi

Working Paper 32341
http://www.nber.org/papers/w32341

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
April 2024

Bank Runs, Fragility, and Regulation
Manuel Amador and Javier Bianchi
NBER Working Paper No. 32341
April 2024
JEL No. E32,E44,E58,G01,G21,G33

ABSTRACT

We examine banking regulation in a macroeconomic model of bank runs. We construct a general equilibrium model where banks may default because of fundamental or self-fulfilling runs. With only fundamental defaults, we show that the competitive equilibrium is constrained efficient. However, when banks are vulnerable to runs, banks' leverage decisions are not ex-ante optimal: individual banks do not internalize that higher leverage makes other banks more vulnerable. The theory calls for introducing minimum capital requirements, even in the absence of bailouts.

Manuel Amador
Department of Economics
University of Minnesota
4-101 Hanson Hall
1925 Fourth Street South
Minneapolis, MN 55455
and Federal Reserve Bank of Minneapolis
and also NBER
amador.manuel@gmail.com

Javier Bianchi
Federal Reserve Bank of Minneapolis
90 Hennepin Avenue
Minneapolis, MN 55401
javieribianchi@gmail.com

1 Introduction

The banking turmoil of March 2023 has once again highlighted banks' vulnerability to runs. As articulated by Diamond and Dybvig (1983), banks' reliance on short-term debt is at the center of their vulnerability. When depositors panic and withdraw funds from a bank, it triggers a liquidity problem that may result in the bank defaulting on its obligations. Runs can be self-fulfilling and bring down banks that would otherwise remain solvent.

The recent turmoil leading to the collapse of Silicon Valley Bank and Signature Bank has sparked a renewed discussion on the need to tighten banking regulation.[1] Although it is commonly agreed that higher equity buffers would make the banking system more resilient, an unsettled question is why banks do not *choose* to hold more capital to reduce their exposure to runs.

In this paper, we provide a theory of banking regulation in a general equilibrium environment where banks may be vulnerable to self-fulfilling runs. Our model elucidates why a social planner would command higher equity requirements, as individual banks fail to internalize that increasing their leverage raises the likelihood of runs on other banks. The crux of the mechanism is as follows: when banks face runs, they are net sellers of assets because they need liquidity to repay the deposits that are being withdrawn. With higher aggregate bank equity, asset prices remain elevated, making it easier for banks to access liquidity during runs. However, because banks do not account for these positive general equilibrium effects, they take on too much leverage from a social point of view.

Our theory builds on Amador and Bianchi (2024), where we developed a macroeconomic model of self-fulfilling bank runs. Banks have limited commitment and may default because of idiosyncratic fundamental shocks or self-fulfilling runs depending on their leverage and asset prices, which are endogenously determined in a Walrasian market. The normative analysis in our previous work takes an initial debt level as given and shows how a credit easing policy can reduce ex post the number of banks facing runs. In this paper, we take an ex-ante macroprudential perspective and analyze the private and social optimal leverage decisions under the risk of a future default. Specifically, we examine the extent to which a social planner would choose a leverage different from the one in the competitive equilibrium.

The model has three periods and a continuum of banks. In the initial period, banks choose capital investment and make equity payout and leverage decisions. The price at which banks borrow in this initial period reflects the probability of a bank's default in the subsequent period. A bank default in the intermediate period can be triggered by a fundamental shock or by a self-fulfilling run, as in Cole and Kehoe (2000). When the bank faces a run, it must sell some of its

[1] See the July 2023 proposal by the Federal Reserve, the Federal Deposit Insurance Corporation (FDIC), and the Office of the Comptroller of the Currency (OCC). For a skeptical perspective, see the July 2023 statement by Governor Christopher J. Waller.

assets to repay the deposits if it does not default. If there exists a positive equilibrium spread between the return on capital and deposits, banks may be vulnerable to runs even though their assets are fully liquid. Finally, if the bank continues operating, it produces in the final period and may again repay or default on the deposits.

Our analysis compares the competitive equilibrium with the solution of a constrained social planner problem that chooses borrowing decisions on behalf of banks in the initial period to maximize banks' ex-ante welfare. In this constrained-efficient allocation, banks retain the ability to default in the intermediate and final periods, and the price of capital clears the capital market in each period. The critical distinction between the constrained-efficient allocation and the competitive equilibrium is that the planner internalizes how the amount of borrowing in the initial period affects the prices of capital and, in turn, affects the continuation values for banks.

In the first part of the paper, we provide welfare theorem-like results in the model where bank runs cannot happen, although banks may still default because of fundamentals.[2] We show that competitive equilibria are constrained efficient, and establish the existence and uniqueness of a competitive equilibrium.[3] To understand the intuition behind the constrained-efficiency result, consider an individual bank's payoff in the competitive equilibrium as a function of its leverage. In the competitive equilibrium, its payoff must be weakly higher than the payoff of choosing the constrained-efficient level of leverage when *facing the competitive equilibrium price of capital*. Now, in our model, repaying banks are neither net buyers nor net sellers of capital. Thus, if the price of capital were to deviate from the one that would clear the market, the bond price must weakly increase. So, the payoff of choosing the constrained-efficient level of leverage when facing the price of capital in the constrained-efficient solution must be weakly lower than the payoff of choosing that same leverage when facing the competitive equilibrium prices. Given that banks' payoffs must be weakly higher in the constrained-efficient allocation than in the competitive equilibrium, the two payoffs must coincide.

In the second part of the paper, we demonstrate that in the presence of bank runs, implementing the constrained-efficient allocation requires a strictly positive tax on leverage. The Euler equation for bank leverage reveals that in the presence of runs, individual banks have incentives to reduce leverage to reduce their exposure to bank runs—this is because runs are costly and higher leverage makes a run more likely. However, the planner internalizes that higher aggregate leverage leads to lower demand for capital in the future and, thus, lower asset prices. Given that the marginal repaying bank becomes a net seller of assets during a run, the reduction in asset prices reduces the

[2] That is, we assume in this section that creditors always coordinate on the good equilibrium.

[3] Models with incomplete markets can be prone to non-existence or multiplicity, and there are only a few available results on existence and uniqueness. Hart (1975) provides a seminal example of non-existence. For a proof of existence in a model with collateralized borrowing, see Araujo, Páscoa and Torres-Martínez (2002). See below for a discussion of the literature.

value from repaying and leaves the bank more vulnerable to a run. Implementing the constrained-efficient solution thus requires a tax on leverage.

Literature. Our paper is related to several strands of the literature. First, our paper is related to a vast literature on bank runs. Following Diamond and Dybvig (1983), the literature has explored the role of various policies, including deposit insurance, suspension of convertibility, bailouts, and lender of last resort (e.g., Cooper and Ross, 1998; Ennis and Keister, 2009; Keister, 2016; Dávila and Goldstein, 2023). In the context of models with a single bank, Diamond and Kashyap (2016) and Kashyap, Tsomocos and Vardoulakis (2024) study increases in capital requirements and the extent to which it is feasible or desirable to make banks run-proof. In contrast to these studies, our paper examines the scope for policies that emerge because of general equilibrium considerations.[4] In particular, we uncover a pecuniary externality that requires tightening banks' leverage constraints.

Our paper is related to the literature analyzing the properties of competitive equilibrium in economies with imperfect financial markets. In the general equilibrium literature with incomplete markets, early examples with constrained inefficiency results are Hart (1975) and Stiglitz (1982). Geanakoplos and Polemarchakis (1985) provide a generic result on constrained inefficiency. In the literature on endogenous debt constraints with complete markets, Kehoe and Levine (1993) consider pure exchange economies and show that competitive equilibria are constrained efficient when there is a single good but not with multiple goods.[5] Our model features both limited commitment and incomplete markets for risk, but our paper is different from these two strands of this literature in that we examine an economy with equilibrium default. In particular, we show that the competitive equilibrium is constrained efficient when defaults are triggered by fundamentals but constrained inefficient when triggered by self-fulfilling runs.

Our paper is also related to the literature on macroprudential policy. In this literature, price-dependent financial constraints give rise to amplification effects that require ex-ante government restrictions on borrowing (e.g., Lorenzoni, 2008; Bianchi, 2011; Stein, 2012).[6] Our environment differs from those in the existing literature by introducing equilibrium default. Crucially, we show how the interplay between equilibrium default and asset prices gives rise to a pecuniary externality that leads to over-leverage in the presence of self-fulfilling runs.

[4] Allen and Gale (2000) and Uhlig (2010) present general equilibrium models of runs that feature contagion effects, but they focus on fundamental runs and do not examine prudential regulations. Egan, Hortaçsu and Matvos (2017) develop and estimate a model with multiple equilibria in deposit rates where defaults are based on fundamentals.

[5] For other related studies in this literature, see, for example, Alvarez and Jermann (2000), Jeske (2006), Bloise and Reichlin (2011), and Martins-Da-Rocha, Phan and Vailakis (2022).

[6] Other examples include Gromb and Vayanos (2002), He and Kondor (2016), Bianchi and Mendoza (2018), Dávila and Korinek (2018), and Kilenthong and Townsend (2021). In addition, Farhi, Golosov and Tsyvinski (2009), Gersbach and Rochet (2012), and DiTella (2019) instead study economies with asymmetric information where incentive compatibility constraints depend on market prices. See also Kurlat (2021), Ottonello, Perez and Varraso (2022) and Lanteri and Rampini (2023).

In a related paper, Gertler, Kiyotaki and Prestipino (2020) examine numerical simulations of how countercyclical capital requirements can reduce vulnerability to banking panics. They build on the Gertler and Kiyotaki (2015) model, where strategic complementarities between banks lead to two equilibria, one with high asset prices and all banks repaying, and one with low asset prices and all banks defaulting. In contrast, in our model, runs occur in individual banks and emerge because of strategic complementarities among their depositors. This distinction is important for the role of short-term debt and policies such as lender of last resort. In addition to these differences in the environment, we also formally characterize the constrained-efficient allocation and solve for optimal macroprudential policies.

Our paper also relates to the literature on endogenous equilibrium default. In the context of a single borrower, the literature has advanced results on existence (Chatterjee and Eyigungor, 2012), uniqueness (Auclert and Rognlie, 2016; Aguiar and Amador, 2019), and efficiency (Aguiar, Amador, Hopenhayn and Werning, 2019). Our paper differs by considering a general equilibrium model with multiple borrowers where endogenous fluctuations in asset prices play a central role.[7]

Outline. Sections 2 and 3 present the model. Section 4 provides a characterization of the economy without runs. Section 5 examines optimal banking regulation in the presence of runs. Section 6 concludes.

2 Model

The economy has three periods, $t \in \{0, 1, 2\}$, and is populated by a continuum of banks and investors, both of measure one. There is a single consumption good produced using capital with a linear technology. We assume that banks have direct access to the production technology, in line with the most recent strands of macro-finance models. Capital does not depreciate, and it is in fixed aggregate supply, equal to K.

Banks. Banks' preferences over a stream of dividend payments are given by

$$u(c_0) + \beta \mathbb{E} u(c_1) + \beta^2 \mathbb{E} u(c_2),$$

where $\beta > 0$ is the discount factor, and $u = \log$.

In period 0, all banks start with b_0 units of maturing debt and capital holdings $k_0 = K$. The banks use the capital to produce zk_0 units of the final good. The value of $z > 0$ is the productivity

[7] Dubey, Geanakoplos and Shubik (2005) provides a proof of existence in a general equilibrium model with endogenous default, but with pooling of assets where the bond price depends on aggregate default rates and not on individual leverage.

of a bank that has not previously defaulted, assumed constant in all periods. Banks can issue one-period bonds to creditors, which promise a payment of $R > 0$ in the subsequent period.[8] A bank chooses dividend payments, c_0, issues new short-term debt, b_1, and chooses a new level capital, k_1, which is purchased in competitive markets. The bank's budget constraint is given by

$$c_0 = (z + p_0)k_0 - Rb_0 + q_0(b_1, k_1)b_1 - p_0 k_1, \tag{1}$$

where p_0 is the price of capital, and q_0 represents the bond price schedule as a function of the portfolio chosen by the bank. We assume that banks cannot default in this initial period.[9]

In period 1, banks may repay or default on their debt. If a bank defaults, it cannot borrow or save in bonds and cannot trade in the market for capital. We also assume that the defaulting bank keeps a fraction of its capital, while the remaining capital is lost. The value of defaulting for a bank in period 1 is

$$V_1^D(k_1, z_1^D) = u(z_1^D k_1) + \beta u(z_2^D k_1), \tag{2}$$

where z_1^D and z_2^D encapsulate the fraction of capital kept after default, or equivalently, the productivity during default. In what follows, we assume that z_1^D is drawn from some distribution with support in $[\underline{z}, \overline{z}] \subset [0, z]$, i.i.d. across banks, while $z_2^D \in (0, z)$ is predetermined and common between banks. We let F denote the c.d.f. of z_1^D and f its density.

If the bank repays its deposits and is able to keep borrowing, its budget constraint is given by

$$c_1 = (z + p_1)k_1 - Rb_1 + q_1(b_2, k_2)b_2 - p_1 k_2, \tag{3}$$

where c_1 is the level of dividends in period 1, p_1 is the price of capital, q_1 is the bond price schedule, and b_2 and k_2 are the new choices of short-term debt and capital. That is, the bank collects the return on its assets, pays its debts, and then issues new bonds and buys new capital. As in period 0, the bond price depends on the portfolio chosen by the bank.

In period 2, the bank can choose again whether to repay or to default. If the bank defaults, its value is given by

$$V_2^D(k_2) = u(z_2^D k_2). \tag{4}$$

If the bank repays, it consumes in the final period the output produced minus the debt repaid with interest. That is,

$$c_2 = zk_2 - Rb_2. \tag{5}$$

[8] Standard reasons why issuing short-term debt may be optimal have to do with liquidity benefits (Stein, 2012) or incentive reasons under asymmetric information (Diamond and Rajan, 2000; Calomiris and Kahn, 1991). We could extend the model to incorporate these considerations, but for simplicity, we take as given that the bank issues short-term debt, as observed in practice.

[9] This can be rationalized with a low enough initial debt b_0.

Investors. Investors are risk neutral and discount future consumption at a rate $1/R$ across two subsequent periods. They buy the bonds supplied by the banks, and do not hold any capital stock.

The Banks' Problem and Bond Prices

We describe the bank's problem starting from the last period.

Period $t = 2$ problem. In the final period, the bank's value is given by

$$V_2(b_2, k_2) = \max_{d_2 \in \{0,1\}} \Big\{(1 - d_2)u(zk_2 - Rb_2) + d_2 u(z_2^D k_2)\Big\}, \tag{6}$$

where d_2 represents the default decision, and the value of repayment reflects that in the final period, the bank consumes the output net of the debt repayment.

The optimal default decision is given by:

$$d_2(b_2, k_2) = \begin{cases} 1 & \text{if } \; zk_2 - Rb_2 < z_2^D k_2, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where we assumed that the bank repays if indifferent.

Given that there is no uncertainty in the final period, the bond price at $t = 1$ equals 1 if $zk_2 - Rb_2 \geq z_2^D k_2$ and 0 otherwise. Letting

$$\phi \equiv z - z_2^D,$$

we can express the bond price schedule as

$$q_1(b_2, k_2) = \begin{cases} 1 & \text{if } \; Rb_2 \leq \phi k_2, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Using that $z_2^D \in (0, z)$—that is, default is costly but not infinitely so—it follows that

$$\phi \in (0, z). \tag{9}$$

Note that a bank with no debt ($b_2 = 0$) or with positive holdings of bonds ($b_2 < 0$) does not default, as $zk_2 - Rb_2 \geq z_2^D k_2$ for all $b_2 \leq 0$.

Period $t = 1$ problem. In period 1, a bank can be subject to a run. Following Cole and Kehoe (2000) and Amador and Bianchi (2024), we assume that the bank issues bonds before a repayment or default decision has been made. The fact that investors are atomistic introduces the possibility of a coordination failure. If investors suddenly panic and become unwilling to roll over the bonds, the bank faces a liquidity problem and may be pushed to default on its debt, which in turn rationalizes investors' initial panic.

We distinguish the value of repayment in the case when the bank faces a run and when it does not. Consider first the case without runs. Define the net worth of the bank as

$$n_1 \equiv (z + p_1)k_1 - Rb_1.$$

Using (3) and (8), we obtain that the value of repayment for the bank in this case is

$$V_1^R(n_1) = \sup_{c_1 \geq 0, k_2 \geq 0, b_2} \Big\{ u(c_1) + \beta u(zk_2 - Rb_2) \Big\}, \tag{10}$$

$$\text{subject to:}$$

$$c_1 = n_1 + b_2 - p_1 k_2,$$

$$Rb_2 \leq \phi k_2.$$

We let $k_2(n_1)$ and $b_2(n_1)$ denote optimal policy functions for capital and debt, respectively. If the constraint set is empty, we set $V_1^R(n_1) = -\infty$.

Notice that the leverage constraint reflects that the bank can borrow at the risk-free rate as long as promised repayments do not exceed ϕk_2, and that a bank would never choose to borrow more than this amount as the bond price would be zero, as implied by (8).

We summarize the value function of the bank's problem in the following lemma.

Lemma 1. *The value function $V_1^R(n_1)$ is such that*

$$V_1^R(n_1) = \begin{cases} \infty & \text{if } p_1 < \phi/R \text{ or } p_1 = \phi/R, n_1 > 0; \\ A + (1+\beta)\log(n_1) + \beta \log(R) + \beta \log\left(\frac{z-\phi}{Rp_1 - \phi}\right) & \text{if } \frac{\phi}{R} < p_1 < \frac{z}{R}, n_1 > 0; \\ A + (1+\beta)\log(n_1) + \beta \log(R) & \text{if } p_1 \geq \frac{z}{R}, n_1 > 0; \\ -\infty & \text{if } p_1 \geq \phi/R, n_1 \leq 0, \end{cases}$$

where $A \equiv \beta \log \beta - (1+\beta)\log(1+\beta)$.

Proof. Consider first the case where $p_1 < \phi/R$. Let k_2 be any finite number. Then, the bank can choose $b_2 = \phi k_2/R$ and $c_1 = n_1 + b_2 - p_1 k_2 = n_1 + (\phi k_2/R - p_1)k_2$. This implies that $V_1^R(n_1) \geq u(n_1 + (\phi/R - p_1)k_2) + \beta u((z-\phi)k_2)$. Given that $\phi/R > p_1$ and $z > \phi$ by (9), it follows that the

$V_1^R(n_1) \longrightarrow \infty$ as $k_2 \longrightarrow \infty$ for any n_1.

The same argument applies when $p_1 = \phi/R$ and $n_1 > 0$. In that case $V_1^R(n_1) \geq u(n_1) + \beta u((z-\phi)k_2)$. Note that in both cases, the value would be less than infinity for finite values of k_2; thus, $k_2 \longrightarrow \infty$ to achieve the optimum.

Consider next the case $p_1 \geq \phi/R$ and $n_1 \leq 0$. Then, from the constraint set, we have that

$$c_1 = n_1 + b_2 - p_1 k_2 \leq n_1 + (\phi/R - p_1)k_2 \leq n_1 \leq 0.$$

where we used that $k_2 \geq 0$. Hence, either the constraint set is empty, or at best, $c_1 = 0$. In both cases, $V_1^R(n_1) = -\infty$.

In the case with $p_1 = z/R$ and $n_1 > 0$, the bank is indifferent between savings or holding capital. In the case where $p_1 > z/R$, the bank strictly prefers to hold bonds, and thus $k_2 = 0$. In both cases, the shape of the value function follows directly from optimization and log utility.

In the case where $\phi/R < p_1 < z/R$ and $n_1 > 0$, the bank's borrowing constraint binds; that is, $Rb_2 = \phi k_2$. The value function can then be solved for using optimization and log utility. □

The first case in the lemma, when the value function is infinite, corresponds to the case where the return on capital is so high that the collateral constraint does not restrict borrowing and investment. In this case, the payoff to the bank is infinite, and can be achieved by ever increasing levels of investment and borrowing. The last case, when the value function is $-\infty$, corresponds to the case where either the bank cannot avoid zero dividends in the first period or where the constraint set is empty.

Let us discuss the second and third cases, which will be the relevant ones in equilibrium. When $p_1 > \phi/R$ and $n_1 > 0$, given log utility, the optimal dividend policy conditional on repayment is

$$c_1(n_1) = \frac{1}{1+\beta} n_1. \tag{11}$$

Because of the absence of risk in period 2, the solution to the portfolio depends on the relative returns of capital and bonds. That is,

$$\frac{\beta}{1+\beta} n_1 = p_1 k_2(n_1) - b_2, \qquad b_2(n_1) \leq \frac{\phi}{R} k_2(n_1), \qquad k_2(n_1) \geq 0. \tag{12}$$

If $p_1 < \frac{z}{R}$, we have that banks borrow up to the limit and choose

$$k_2(n_1) = \frac{\beta}{1+\beta}\left(\frac{n_1}{p_1 - \frac{\phi}{R}}\right), \qquad b_2(n_1) = \frac{\phi}{R} k_2(n_1).$$

If the return on capital is instead equal to the return on debt, $p_1 = \frac{z}{R}$, the bank is indifferent between bonds and capital and can choose any portfolio as long as it is consistent with its consumption policy (11) and the leverage constraint. These optimal policies then explain the value function in the second and third cases of the lemma.

Finally, if $\frac{z}{p_1} < R$, we have that banks do not invest in capital and save $b_2(n_1) = -\frac{\beta}{1+\beta}n_1$.

Runs. Consider now the case where the bank faces a run and cannot borrow but still chooses to repay. A bank that repays when facing a run pays back its maturing debt and decides how much capital to buy and how much to consume using only its available net worth. The payoff is given by

$$V_1^{Run}(n_1) = \sup_{c_1 \geq 0, k_2 \geq 0, b_2} \Big\{ u(c_1) + \beta u(zk_2 - Rb_2) \Big\}, \tag{13}$$

subject to:

$$c_1 = n_1 + b_2 - p_1 k_2,$$

$$b_2 \leq 0.$$

The continuation payoff reflects that the bank does not default in period 2 (as it carries no debt) and consumes the return on its investments. Note that the only difference with (10) is the last inequality constraint in (13): the bank cannot borrow when facing a run.

Again, we let $V_1^{Run}(n_1) = -\infty$ if the constraint set is empty (which occurs when $n_1 < 0$). For $n_1 \geq 0$ the optimal consumption policy is

$$c_1^{Run}(n_1) = \frac{1}{1+\beta} n_1,$$

and the demand for capital depends on the return to capital and R. If $p_1 < \frac{z}{R}$, the bank chooses to invest only in capital. If $p_1 > \frac{z}{R}$, the bank chooses to invest only in bonds. Finally, for $p_1 = \frac{z}{R}$, the bank is indifferent. The payoff function for a bank that repays under a run is then:

$$V_1^{Run}(n_1) = \begin{cases} A + (1+\beta)\log(n_1) + \beta \log(z/p_1) & \text{if } p_1 < \frac{z}{R}, n_1 > 0, \\ A + (1+\beta)\log(n_1) + \beta \log(R) & \text{if } p_1 \geq \frac{z}{R}, n_1 > 0, \\ -\infty & \text{if } n_1 \leq 0. \end{cases} \tag{14}$$

Default thresholds. We can use the value functions to characterize default thresholds: the levels of productivity (under default) that make a bank indifferent between repaying and defaulting. These thresholds depend on whether the bank is facing a run or not.

Let us start by considering a situation without runs. Consider a bank with net worth n_1 and capital k_1. The bank defaults for values of z_1^D such that $V_1^D(k_1, z_1^D) > V_1^R(n_1)$; and repays otherwise. Hence, there exists a threshold $\hat{z}^F$ such that a bank that draws a default productivity higher than

$\hat{z}^F$ defaults, and a bank that draws a default productivity weakly lower than $\hat{z}^F$ repays. If $k_1 = 0$, then $V_1^D(k_1, z_1^D) = -\infty$, and we set $\hat{z}^F = \infty$. For $k_1 > 0$, the threshold is given by

$$\hat{z}^F(n_1, k_1) = \begin{cases} \infty & \text{if } p_1 < \phi/R \text{ or } p_1 = \phi/R, n_1 > 0; \\ e^A \left(\frac{n_1}{k_1}\right)^{1+\beta} \left(\frac{R}{Rp_1 - \phi}\right)^{\beta} & \text{if } \frac{\phi}{R} < p_1 < \frac{z}{R}, n_1 > 0; \\ e^A \left(\frac{n_1}{k_1}\right)^{1+\beta} \left(\frac{R}{z - \phi}\right)^{\beta} & \text{if } p_1 \geq \frac{z}{R}, n_1 > 0; \\ 0 & \text{if } p_1 \geq \phi/R, n_1 \leq 0. \end{cases} \tag{15}$$

We call $\hat{z}^F$ the "fundamental default threshold."

Consider now the case where the bank faces a run. A bank with net worth n_1 and capital k_1 defaults for values of z_1^D such that $V_1^D(k_1, z_1^D) > V_1^{Run}(n_1)$; and repays otherwise. If $k_1 = 0$, then $V_1^D(k_1, z_1^D) = -\infty$, and we set $\hat{z}^{Run} = \infty$. For $k_1 > 0$, the threshold in case of a run is given by

$$\hat{z}^{Run}(n_1, k_1) = \begin{cases} e^A \left(\frac{n_1}{k_1}\right)^{1+\beta} \left(\frac{z/p_1}{z - \phi}\right)^{\beta} & \text{if } p_1 < \frac{z}{R}, n_1 > 0; \\ e^A \left(\frac{n_1}{k_1}\right)^{1+\beta} \left(\frac{R}{z - \phi}\right)^{\beta} & \text{if } p_1 \geq \frac{z}{R}, n_1 > 0; \\ 0 & \text{if } n_1 \leq 0. \end{cases} \tag{16}$$

Whenever a bank with net worth n_1 and capital $k_1 > 0$ draws a default productivity $z_1^D \leq \hat{z}^{Run}(n_1, k_1)$, the bank repays even if it is subject to a run. We call $\hat{z}^{Run}$ the "run threshold."

Note that $\hat{z}^F(n_1, k_1) \geq \hat{z}^{Run}(n_1, k_1)$. Combining (15) and (16), we obtain $\hat{z}^F(n_1, k_1) > \hat{z}^{Run}(n_1, k_1)$ if and only if $k_1 > 0$ and either

$$\text{(i) } p_1 < \frac{z}{R} \text{ and } n_1 > 0; \text{ or (ii) } p_1 < \frac{\phi}{R}. \tag{17}$$

In both (i) and (ii), the return to capital, z/p_1, is strictly higher than the cost of borrowing, R. When $p_1 < z/R$, a bank with positive net worth can leverage ($\phi > 0$) when not facing a run and thus makes a profit from the difference in returns. Thus, the fundamental default threshold is strictly higher than the run threshold.[10] When $p_1 < \phi/R$, this mechanism is even stronger, as a

[10] As we discussed in Amador and Bianchi (2024), in this model, banks can be subject to runs despite having only liquid assets in their balance sheets, a feature that resonates with the failure of Silicon Valley Bank (see e.g., Metrick, 2024). When the return on capital is higher than the return on bonds, a bank that can borrow achieves a higher profit by exploiting the interest rate margin. If a bank faces a run, however, it experiences a loss in franchise value, which can leave it more prone to default and, therefore, vulnerable to a self-fulfilling run. This feature of our model is also present in Drechsler, Savov, Schnabl and Wang (2023) and Jiang, Matvos, Piskorski and Seru (2023) who explore how higher interest rates may generate simultaneously higher franchise value and lower asset values, thus leaving banks more vulnerable to runs. Abstracting from runs, Di Tella and Kurlat (2021) find that the optimal hedging strategy by banks is consistent with long-duration assets and short-duration liabilities, thus implying that in equilibrium banks take losses when interest rates rise.

repaying bank that is not facing a run can leverage without limits, independently of its net worth. As we will see below, this latter case is not compatible with general equilibrium.

A bank with net worth n_1 and capital k_1 that draws a default productivity z_1^D that is higher than $\hat{z}^F(n_1, k_1)$ always defaults. If that bank draws a default productivity value z_1^D weakly lower than $\hat{z}^{Run}(n_1, k_1)$, then it repays. However, the behavior of the bank when it draws a default productivity value z_1^D between $\hat{z}^{Run}(n_1, k_1)$ and $\hat{z}^F(n_1, k_1)$ is indeterminate. In this case, we say that the bank is *vulnerable* to a run. If investors expect the bank with such a productivity level to repay, then they are willing to lend, and the bank repays. If investors expect such a bank to default, then the bank cannot borrow and defaults. In both cases, the investors' beliefs are confirmed by the equilibrium behavior of the bank. The issue here is a coordination failure between individual bank investors, as in Cole and Kehoe (2000) or Diamond and Dybvig (1983).

As in Cole and Kehoe (2000), we assume that a sunspot determines the outcome when a bank is vulnerable to a run.[11] Specifically, with probability λ, a run happens, and the vulnerable bank defaults. With complement probability $1 - \lambda$, a run does not happen, and the vulnerable bank is able to borrow. We assume that this sunspot draw is i.i.d. across banks.

Incorporating this, we let $d_1(n_1, k_1, z_1^D)$ denote the probability that an individual bank with net worth n_1, capital k_1, and default productivity z_1^D will default in period $t = 1$. Then,

$$d_1(n_1, k_1, z_1^D) = \begin{cases} 0 & \text{if } z_1^D \leq \hat{z}^{Run}(n_1, k_1), \\ \lambda & \text{if } \hat{z}^{Run}(n_1, k_1) < z_1^D \leq \hat{z}^F(n_1, k_1), \\ 1 & \text{if } z_1^D > \hat{z}^F(n_1, k_1), \end{cases} \tag{18}$$

Because investors are assumed to be risk neutral, the bond price in period 0 is given by the expected value of repayment in period 1. That is, abusing notation, we see that

$$q_0(n_1, k_1) = (1 - \lambda)F(\hat{z}^F(n_1, k_1)) + \lambda F(\hat{z}^{Run}(n_1, k_1)). \tag{19}$$

It is worth noting that a bank that has no leverage does not default and is not vulnerable to a run, a result that arises from the fact that default is costly (and it is feasible for a bank to not borrow and receive a payoff strictly higher than the default value). The following lemma formalizes this.

[11] This is a somewhat arbitrary equilibrium selection device, and alternative assumptions are possible. See, for example, the specification in Bocola and Dovis (2019).

Lemma 2. *A bank with no debt never defaults. That is,*

$$\hat{z}^F(n_1, k_1) \geq \hat{z}^{Run}(n_1, k_1) > z$$

for $n_1 = (z + p_1)k_1 - Rb_1$ *and* $b_1 \leq 0$*.*

Proof. We have already argued that $\hat{z}^F(n_1, k_1) \geq \hat{z}^{Run}(n_1, k_1)$. So we just need to show that $z^{Run}(n_1, k_1) > z$. Note that if $k_1 = 0$, the result is immediate, as $\hat{z}^{Run}(n_1, 0) = \infty$. So consider $k_1 > 0$. Given $b_1 \leq 0$, it follows that $n_1 > 0$.

First, note that by (16), $\hat{z}^{Run}(n_1, k_1) \geq \hat{z}^{Run}((z + p_1)k_1, k_1)$. Thus, it is enough to show that $\hat{z}^{Run}((z + p_1)k_1, k_1) > z$.

When $p_1 < z/R$, $\hat{z}^{Run}((z + p_1)k_1, k_1) > z$ is equivalent to

$$e^A (z + p_1)^{1+\beta} \left(\frac{z/p_1}{z - \phi}\right)^\beta \frac{1}{z} > 1. \tag{20}$$

Using that $\phi > 0$, we have that

$$e^A (z + p_1)^{1+\beta} \left(\frac{z/p_1}{z - \phi}\right)^\beta \frac{1}{z} > e^A (z + p_1)^{1+\beta} \left(\frac{1}{p_1}\right)^\beta \frac{1}{z} \equiv H(p_1).$$

Note that

$$H'(p_1) = e^A \frac{1}{z} \frac{(z + p_1)^\beta}{p_1^{1+\beta}} (p_1 - \beta z).$$

And thus, $H'(p_1) < 0$ for $0 < p_1 < \beta z$, $H'(p_1) > 0$ for $p_1 > \beta z$, and $H'(p_1) = 0$ for $p_1 = \beta z$. Hence, H achieves a minimum for $p_1 \geq 0$ at $p_1 = \beta z$. Thus, $H(p_1) \geq H(\beta z) = 1$, where the last equality follows from substitution and manipulation of H. Thus, (20) holds.

When $p_1 \geq z/R$, $\hat{z}^{Run}((z + p_1)k_1, k_1) > z$ becomes

$$e^A (z + p_1)^{1+\beta} \left(\frac{R}{z - \phi}\right)^\beta \frac{1}{z} > 1.$$

Similarly, then

$$e^A (z + p_1)^{1+\beta} \left(\frac{R}{z - \phi}\right)^\beta \frac{1}{z} > e^A (z + p_1)^{1+\beta} \left(\frac{R}{z}\right)^\beta \frac{1}{z} \geq e^A (z + p_1)^{1+\beta} \left(\frac{1}{p_1}\right)^\beta \frac{1}{z} = H(p_1),$$

where the first inequality uses $\phi > 0$, and the second uses that $R/z \geq p_1$. Then, $H(p_1) \geq 1$ delivers the result. □

From this it follows immediately that $q_0(n_1, k_1) = 1$ for $n_1 = (z + p_1)k_1 - Rb_1$ and $b_1 \leq 0$.

Period $t = 0$ problem. The problem of a bank at $t = 0$ consists of choosing its dividend, level of borrowing, and capital investments to maximize its payoff. Unlike the $t = 1$ problem, the bank problem in period $t = 0$ faces uncertainty, in particular, regarding the outside option shock z_1^D and the sunspot at $t = 1$. This implies that the bank may now choose a portfolio that implies a positive default probability in period 1 and investors will adjust the bond price they offer accordingly.

The problem of a bank at time $t = 0$ can be written as follows:

$$
\begin{aligned}
V_0(n_0) = \max_{c_0 \geq 0, k_1 \geq 0, b_1} & \Bigg\{ u(c_0) \\
& + \beta \int_{\underline{z}}^{\overline{z}} \Big[d_1(n_1, k_1, \tilde{z}) V_1^D(k_1, \tilde{z}) + (1 - d_1(n_1, k_1, \tilde{z})) V_1^R(n_1) \Big] dF(\tilde{z}) \Bigg\}, \\
& \text{subject to} \\
& \qquad c_0 = n_0 + q_0(n_1, k_1) b_1 - p_0 k_1, \\
& \qquad n_1 = (z + p_1) k_1 - R b_1,
\end{aligned}
\tag{21}
$$

where $n_0 = (z + p_0)k_0 - Rb_0$.

A reader may wonder why the value function of a bank that repays when facing a run, V_1^{Run}, does not appear in (21). The reason is that V_1^{Run} represents the payoff of an off-equilibrium choice, as a bank that repays in the presence of a run will not face one. Although not appearing directly in (21), V_1^{Run} is nonetheless essential for the determination of the run threshold, and thus for the default probabilities and the equilibrium price schedule faced by the bank.

We let $k_1(n_0)$, $c_0(n_0)$ and $b_1(n_0)$ denote optimal choices of capital, dividends, and borrowing, respectively, as a function of the initial net worth.

3 Competitive Equilibrium

We have stated the problem of individual banks and investors given prices for capital p_0, p_1. We now discuss how these prices are determined in general equilibrium.

In what follows, we narrow attention to *symmetric pure-strategy equilibria*: one where banks with the same state variables choose the same policies in all periods, and such policies do not involve mixed strategies. The market clearing condition for capital requires that banks' demand for capital equals K in periods $t = 0$ and $t = 1$. We can define a (symmetric, pure-strategy) competitive equilibrium as follows.

Definition 1. Given an initial debt level, B_0, and a run probability, λ, a *competitive equilibrium* consists of prices of capital in the two periods, $\{p_0, p_1\}$, borrowing choices in the two periods for repaying banks, $\{B_1, B_2\}$, capital choices in the two periods for repaying banks, $\{K_1, K_2\}$, initial net worth in the two periods $\{N_0, N_1\}$, default threshold functions $\{\hat{z}^F, \hat{z}^{Run}\}$, a default probability function d_1, a period-1 repayment value function V_1^R, a period-1 default value function V_1^D, and a price schedule q_0 such that the following conditions obtain:

(a) Non-defaulting banks optimize.

For $t = 0$, given $\{p_0, q_0, V_1^R, V_1^D, d_1\}$, there exist optimal policy functions $\{k_1, b_1\}$ that solve the bank's problem, (21). In addition, $K_1 = k_1(N_0)$ and $B_1 = b_1(N_0)$.

For $t = 1$, given p_1, there exist optimal policy functions $\{k_2, b_2\}$ that solve the repaying bank's problem (10) where V_1^R is the resulting value function.

In addition, $K_2 = k_2(N_1)$ and $B_2 = b_2(N_1)$.

(c) Default is optimal. That is, given p_1 and λ, V_1^D is given by (2); the default threshold functions $\{\hat{z}^F, \hat{z}^{Run}\}$ are given by (15) and (16), respectively; and the default probability function d_1 is given by (18).

(b) Investors' pricing condition holds. That is, the bond price schedule q_0 satisfies (19) given λ and the default threshold functions.

(d) The market for capital clears at $t = 0$ and $t = 1$. That is, $K_1 = K$ and

$$\left[(1-\lambda)F(\hat{z}^F(N_1, K_1)) + \lambda F(\hat{z}^{Run}(N_1, K_1))\right](K_2 - K) = 0,$$

and where $N_0 = (z + p_0)K - RB_0$ and $N_1 = (z + p_1)K_1 - RB_1$.

The last market clearing condition in part (d) ensures that the total demand for capital equals the total supply of capital. As long as a positive mass of banks does not default in period $t = 1$, the condition requires that $K_2 = K$.

A first result is that a certain default in period $t = 1$ is not part of an equilibrium: a bank could instead choose not to borrow at all in period $t = 0$.

Lemma 3. *In any competitive equilibrium,* $q_0(N_1, K_1) > 0$.

Proof. Toward a contradiction, suppose that N_1, K_1 is such that $q_0(N_1, K_1) = 0$. This requires that $B_1 > 0$, as otherwise $q_0(N_1, K_1) = 1$. For the bank problem in period $t = 0$ to have a solution and for $K_1 = K > 0$, we need that

$$N_0 > 0,$$

The bank's equilibrium payoff at $t = 0$ is:

$$V_0 = \log(N_0 - K_1) + \beta \int_{\underline{z}}^{\overline{z}} V_1^D(K_1, \tilde{z}) dF(\tilde{z}).$$

Consider then the alternative where the bank chooses $B_1 = 0$ and $B_2 = 0$ and chooses the same K_1 and $K_2 = K_1$. This strategy is feasible, and the $t = 0$ payoff to the bank is:

$$\hat{v}_0 = \log(N_0 - K_1) + \beta \left[\log(zK_1) + \beta^2 \log(zK_1)\right].$$

But we have that $z > \overline{z}$, and thus $\hat{v}_0 > V_0$, a contradiction of banks' optimality. □

The above implies that in an equilibrium, there will be a strictly positive fraction of banks that do not default in period $t = 1$. With this, we can show that the price of capital at $t = 1$ is such that $\phi/R \leq p_1 \leq z/R$.

Lemma 4. *In any competitive equilibrium, $p_1 \in (\phi/R, z/R]$.*

Proof. If $p_1 < \phi/R$, then repaying banks in period $t = 1$ (of which there is a positive mass of them) attain an infinite payoff while demanding an infinite amount of capital, a violation of the market clearing condition $K_2 = K < \infty$.

If $p_1 > z/R$, then repaying banks in period $t = 1$ demand zero capital, a violation of the market clearing condition $K_2 = K > 0$.

Suppose now that $p_1 = \phi/R$. Then by Lemma 1, it follows that $N_1 > 0$, or else $V_1^R = -\infty$ and all banks will default in period $t = 1$ (as $K_1 > 0$ by market clearing). But if $p_1 = \phi/R$ and $N_1 > 0$, the banks attain an infinite value in period $t = 0$ while demanding infinite capital, a violation of market clearing. □

Lemmas 3 and 4 imply that in equilibrium, $N_1 > 0$.

3.1 The Initial Choice of Capital and Leverage

We have not yet discussed the choice of borrowing and capital in period $t = 0$, b_1, and k_1— that is, the solution to Problem (21). We know that in period $t = 0$, equilibrium requires that the demand for capital be $K_1 = K > 0$. Let us focus on the case where $k_1 > 0$, and define leverage l_1 to be

$$l_1 \equiv \frac{b_1}{k_1}.$$

In an equilibrium, not all banks default in period $t = 1$, and as a result, value functions and demand for capital in period $t = 1$ are bounded (Lemmas 3 and 4). Therefore, inspection of (15) and (16) shows that the default thresholds that are relevant in equilibrium are just functions of n_1/k_1, or alternatively of just l_1.[12]

Redefining functions. Abusing notation, we let $\hat{z}^F(l_1|p_1)$ and $\hat{z}^{Run}(l_1|p_1)$ refer to the thresholds as functions of leverage, where we make the dependence on the period 1 price of capital explicit. It is then immediate from equation (18) that the default probability function is also a function of leverage, which we redefine by $d_1(l_1, z_1^D|p_1)$, again making the dependence on p_1 explicit. Finally, the price function for the bonds in period $t = 0$, given by (19), is also just a function of a bank's leverage, which we denote by $q_0(l_1|p_1)$. Similarly, we let $V_1^R(n_1|p_1)$ denote the repayment value function in period $t = 1$, where again we make the dependence on the period 1 price of capital explicit.

Optimality at $t = 0$. We can now analyze the bank's problem in period $t = 0$. That is, (21). We have the following result.

Lemma 5. *In a competitive equilibrium, there is no l_1 such that $q_0(l_1|p_1)l_1 \geq p_0$.*

Proof. Suppose that a bank in period $t = 0$ chooses $l_1 = l_1^\star$ and $k_1 = K$. If $q_0(l_1^\star|p_1)l_1^\star \geq p_0$, then the bank can increase its demand for capital without changing c_0. This increases its continuation value without bounds as the default probability at $t = 1$ is less than one (as $q_0(l_1^\star|p_1) > 0$); thus, $k_1 = K$ is not optimal.

Similarly, if the bank chooses a $l_1^\star$ such that $q_0(l_1^\star|p_1)l_1^\star < p_0$, then the bank can instead choose l_1 such that $q_0(l_1|p_1)l_1 \geq p_0$ without changing its consumption in period $t = 0$. As was the case above, the bank can then increase k_1 without limits and thus increase its continuation value without bounds.

Taken together, the demand for capital in period $t = 0$ is not bounded if $q_0(l_1|p_1)l_1 \geq p_0$ for some l_1, and thus cannot be part of a competitive equilibrium. □

It follows from this lemma that, in a competitive equilibrium, initial net worth must be positive. That is, $n_0 \geq 0$, as otherwise the feasible set in period $t = 0$ is empty, and the bank's problem in period $t = 0$ is not well defined. We will restrict attention to competitive equilibria with $n_0 > 0$,

[12] To see this, note that $n_1/k_1 = ((z + p_1)k_1 - Rl_1k_1)/k_1 = z + p_1 - Rl_1$.

which guarantees strictly positive consumption in period $t = 0$, and thus a payoff to the banks that is bounded below. Given a choice of leverage l_1, we can solve for the optimal capital and consumption choice in period $t = 0$:

$$c_0(n_0) = \frac{1}{1+\beta(1+\beta)} n_0, \tag{22}$$

$$k_1(n_0, l_1|p_0, p_1) = \frac{\beta(1+\beta)}{1+\beta(1+\beta)} \left(\frac{n_0}{p_0 - q_0(l_1|p_1) l_1} \right). \tag{23}$$

Despite facing default risk, the optimal policies for consumption and capital follow a simple expression, owing to log utility and the fact that the probability of default is only a function of leverage.

When choosing leverage, a key consideration is how leverage affects the bond price at which the bank can issue bonds, $q_0(l_1|p_1)$, given by (19). Because $\hat{z}^F$ and $\hat{z}^{Run}$ are weakly decreasing in l_1, when solving (21), the bank takes into account that higher leverage reduces the price at which it can sell bonds.

We can glimpse at the implications of this by looking at a necessary condition for the optimality of l_1. But before showing this, we will use the following assumption, which guarantees the differentiability of the bank's problem.

Assumption 1. *The probability distribution function f is continuous and such that $f(\underline{z}) = f(\overline{z}) = 0$.*

We are ready to state the following result.

Lemma 6 (Necessary condition for l_1). *Suppose that $n_0 > 0$ and that Assumption 1 holds. Then, a level of leverage l_1 that solves Problem (21) must satisfy*

$$\frac{1}{c_0(n_0)} - \frac{\beta R}{c_1(n_1)} = -\frac{\overbrace{(1-\lambda) f(\hat{z}^F(l_1|p_1)) \frac{\partial \hat{z}^F(l_1|p_1)}{\partial l_1} + \lambda f(\hat{z}^{Run}(l_1|p_1)) \frac{\partial \hat{z}^{Run}(l_1|p_1)}{\partial l_1}}^{\text{Marginal change in default probability}}}{(1-\lambda) F(\hat{z}^F(l_1|p_1)) + \lambda F(\hat{z}^{Run}(l_1|p_1))} \frac{l_1}{c_0(n_0)}$$

$$- \frac{\overbrace{\lambda f(\hat{z}^{Run}(l_1|p_1)) \frac{\partial \hat{z}^{Run}(l_1|p_1)}{\partial l_1}}^{\text{Marginal change in run probability}}}{(1-\lambda) F(\hat{z}^F(l_1|p_1)) + \lambda F(\hat{z}^{Run}(l_1|p_1))} \frac{\beta}{k_1} \Big[\overbrace{V_1^R(n_1|p_1) - V_1^D(k_1, \hat{z}^{Run}(l_1|p_1))}^{\text{Loss in case of runs}} \Big], \tag{24}$$

where $n_1 = ((z + p_1) - Rl_1)k_1$ and $k_1 = k_1(n_0, l_1|p_0, p_1)$.

Proof. The objective function of the bank's problem in period $t = 0$ is (ignoring the dependence on p_1 for notational simplicity) given by:

$$\log(n_0 + q_0(l_1)l_1k_1 - p_0k_1) + \beta F(\hat{z}^{Run}(l_1))V_1^R(n_1) \\ + \beta \int_{\hat{z}^{Run}(l_1)}^{\hat{z}^F(l_1)} [(1-\lambda)V_1^R(n_1) + \lambda V_1^D(k_1, \tilde{z})]f(\tilde{z})d\tilde{z} + \beta \int_{\hat{z}^F(l_1)}^{\infty} V_1^D(k_1, \tilde{z})f(\tilde{z})d\tilde{z},$$

The thresholds $\hat{z}^F$ and $\hat{z}^{Run}$ are differentiable in l_1, and so is the bond-price function $q_0(l_1)$ given that $f(\underline{z}) = f(\overline{z}) = 0$. It follows that the objective function is differentiable in l_1.

The choice of leverage l_1 must be strictly less than $(z + p_1)/R$ for the bank to have a positive net worth in period 1. If this were not the case, the bank would default for sure in period 1, and we know from the argument in the proof of Lemma 3 that this cannot be optimal. That is, any optimal choice of leverage l_1 is such that $\hat{z}^F(l_1) > \underline{z}$.

Any choice of leverage also must lead to strictly positive consumption in period 0, as otherwise the bank's payoff is $-\infty$ and dominated by $l_1 = 0$ for some $k_1 > 0$, given $n_0 > 0$. It follows that the choice of l_1 must be interior, and the first-order condition with respect to l_1 leads to the condition in the Lemma. □

The bank's Euler equation (24) reflects that when the bank changes its leverage, it brings more resources to the present period, promises future repayments, and changes the probability with which it will default, which affects, in turn, the price at which it borrows. If leverage did not affect default risk at the margin, the right-hand side of (24) would be zero. In that case, the optimal leverage would equate the marginal benefits from increasing consumption today and lowering consumption tomorrow in repaying states. Given that bond prices are actuarially fair, this delivers a standard inter-temporal Euler equation with an intertemporal price given by R. But even in that case, the *level* of default risk matters for leverage, as default is *costly* for banks, and thus there are incentives for banks to reduce borrowing.

The incentives to reduce leverage are easier to see when the choice of leverage has a marginal effect on default risk. In that case, there is a strictly positive wedge between the marginal utility from borrowing and consuming today and reducing consumption tomorrow in repayment states. The wedge on the right-hand side of (24) emerges because when the bank increases leverage, it lowers the default threshold and thus increases the probability of default at the margin. As the likelihood of default increases, the bank must issue the infra-marginal unit of bonds at a lower price, as reflected in the first term.

When default risk is driven solely by fundamental factors, a change in the default threshold due to a marginal increase in leverage does not alter the bank's expected continuation value. This is because, at the threshold, a bank is indifferent between defaulting and repaying in the absence of runs. Therefore, the cost of raising leverage is driven solely by the reduction in the bond price.

However, when a bank is subject to runs, a fall in the default threshold generates a discrete drop in the continuation value for a bank. This is because, at the threshold, the bank would strictly prefer to repay if it could roll over the deposits. This additional cost resulting from leverage is reflected in the last term of (24).

3.2 Equilibrium Characterization Using Leverage

Next, we show that there exists an equilibrium relationship between the aggregate level of borrowing in period $t = 0$ per unit of capital (i.e., the aggregate leverage, $L_1 \equiv B_1/K$) and the price of capital p_1. To see this, let us first look at the cases where $z/R > p_1 > \phi/R$. Then, investment in capital is given by

$$K_2 = \frac{\beta}{(1+\beta)(p_1 - \phi/R)} N_1 = K,$$

which implies that $p_1 = \beta z + (1+\beta)\phi/R - \beta R L_1$. Given that $\phi/R < p_1 < z/R$, it follows that $L_1 \in (\hat{L}, \overline{L})$, where $\hat{L} \equiv ((1+\beta)\phi - z(1-\beta R))/(\beta R^2)$ and $\overline{L} \equiv (z + \phi/R)/R$.

If $z/R = p_1$, a bank is indifferent between investing in capital and bonds, as long as it is consistent with its consumption function and the borrowing constraint. Using (11), (12), and imposing that $K_2 = K$, we can see that it follows that $(p_1 - \phi/R)K \leq \beta N_1/(1+\beta)$, which is equivalent to $L_1 \leq \hat{L}$.

Taken together, in equilibrium, the price of capital p_1 is such that:

$$p_1 = \mathcal{P}_1(L_1) \equiv \begin{cases} \frac{z}{R} & \text{if } L_1 \leq \hat{L}, \\ \beta z + (1+\beta)\frac{\phi}{R} - \beta R L_1 & \text{if } L_1 \in (\hat{L}, \overline{L}). \end{cases} \tag{25}$$

As a function of equilibrium leverage L_1, the price of capital in period 1 is continuous, but features a kink at the level where the leverage constraint becomes binding, which we denote by $\hat{L}$. For $L < \hat{L}$, the leverage constraint does not bind, and arbitrage requires that the return on capital equal the return on bonds, which pins down the price. For $L_1 > \hat{L}$, the price of capital is decreasing in aggregate leverage, and the return on capital exceeds the return on bonds.

The value of L_1 is sufficient to characterize the rest of a competitive equilibrium. To see this, note that given a value for the aggregate leverage L_1, it follows that p_1 is determined by (25). The debt threshold functions, $\hat{z}^F$ and $\hat{z}^{Run}$ are determined by equations (15) and (16), given p_1. This implies that the default probability function d_1 and the bond price schedule q_0 are also determined. Given this, the value of V_1^R is determined by (10). Equation (12) then determines B_2, given that $K_2 = K$ and N_1 is obtained given p_1. The value of V_1^D is given by (2). Finally, given $l_1 = L_1$, equation

(23) determines p_0 as $k_1(N_0) = K$, q_0 is a known function, and $N_0 = (z + p_0)K - RB_0$. Specifically,

$$p_0 = \mathcal{P}_0(L_1) \equiv (1 + \beta(1 + \beta))q_0(L_1|\mathcal{P}_1(L_1))L_1 + \beta(1 + \beta)\left[z - \frac{RB_0}{K}\right]. \tag{26}$$

The only thing left to check for an equilibrium is that L_1 corresponds to banks' optimal choice of leverage given the prices they face.

4 Equilibrium and Constrained Efficiency Without Runs

In this section, we analyze the properties of the competitive equilibrium without runs. That is, we set $\lambda = 0$. We will establish the existence and uniqueness of a competitive equilibrium and show that the economy without runs is constrained efficient. After providing these welfare theorem results, we return to the economy with runs in Section 5 and show that the competitive equilibrium is constrained inefficient.

4.1 Constrained Efficiency Without Runs

We consider the problem of a planner that chooses leverage in period 0 on behalf of banks, leaves all other choices unrestricted, and lets all markets clear competitively. That is, banks retain the ability to make default decisions, and therefore, the bond price at which the planner issues bonds in period 0 continues to reflect the probability of default in period 1, characterized above. As we discussed at the end of the previous section, leverage is all that is needed to characterize an equilibrium. Given the choice of leverage by the planner, the investment decision of banks at $t = 0$ is given by equation (23). General equilibrium then requires that p_0 solves (26).

We assume that the planner maximizes the welfare of banks. Note, however, that because investors have linear utility, they always break even ex ante and therefore remain indifferent among any level of leverage chosen by the planner. The planning problem is

$$\max_{L_1, c_0} \left\{ u(c_0) + \beta \int_{\underline{z}}^{\overline{z}} \max\{V_1^R(n_1|p_1), V_1^D(K, \tilde{z})\} dF(\tilde{z}) \right\}, \tag{27}$$

subject to:

$$0 \le c_0 \le zK - RB_0 + q_0(L_1|p_1)L_1K,$$

$$L_1 < \overline{L},$$

and where:

$$n_1 = (z + p_1)K - RL_1K, \ p_1 = \mathcal{P}_1(L_1).$$

In formulating (27), we have used that the default decision is just a comparison between the equilibrium value function of repayment versus default, given that there are no runs. In addition, we have set the budget constraint as an inequality (at a solution, it will always be binding).

The problem is similar to that of an individual bank, with two key differences. First, the planner is subject to a resource constraint in period 0 that reflects that banks must hold K units of capital.[13] Second, when choosing leverage, the planner internalizes that this affects the price of capital in period 1. In turn, the price of capital affects the default thresholds in period 1 and, as a result, the price at which the planner borrows in period 0.

If we substitute n_1 and p_1 into the objective function as functions of L_1, it is possible to show that the resulting $V_1^R(n_1|p_1)$ and $\hat{z}^F(L_1|p_1)$ functions are continuous and differentiable with respect to L_1. In addition, as $L_1 \nearrow \overline{L}$, $V_1^R(n_1|p_1)$ tends to minus infinity and thus default occurs with certainty at the limit (that is, $\hat{z}^F(L_1|p_1)$ tends to zero). Thus, banks raise no revenue in period $t = 0$ and are certain to default in period $t = 1$ as $L_1 \nearrow \overline{L}$. This means that we can enlarge the constraint set to be $L_1 \leq \overline{L}$ (now with a weak inequality), as the boundary choice of $L_1 = \overline{L}$ is dominated by the choice of $L_1 = 0$. One can show as well that $L_1 \geq \min\{-(z - RB_0/K), 0\}$ as for $L_1 \leq 0$, $q(L_1|p_1) = 1$. Note that q is a bounded transformation of $\hat{z}^F$, and thus the set of (c_0, L_1) that satisfies the constraints is bounded. This set is also closed as all the functions involved are continuous and the inequalities are all weak. Finally, the objective function is continuous in c_0 and L_1 and is also bounded everywhere except at $c_0 = 0$ (where it takes a minus infinity value).

We will impose the following assumption:

Assumption 2. *The parameters are such that* $zK - RB_0 + \overline{m} > 0$ *where* $\overline{m} \equiv \max_{L_1 \leq \overline{L}} q_0(L_1|\mathcal{P}_1(L_1))L_1K$.

This assumption guarantees that there exists a feasible choice (c_0, L_1) that strictly dominates any allocation with $c_0 = 0$.[14] Thus, under this condition, there exists a solution to the planner's problem. We define such solutions as representing constrained-efficient levels of leverage.

Definition 2 (Constrained-efficient leverage)**.** In the case without runs ($\lambda = 0$), we say that a competitive equilibrium is constrained efficient if the level of leverage L_1 solves the planning problem (27).

We can use Assumption 1, which guarantees differentiability of the objective function, to state

[13] Notice that p_0 does not appear in the constraints of the planner, because in period 0, banks are neither net buyers nor net sellers of capital.

[14] A simpler sufficient condition for Assumption 2 is that $zK - RB_0 > 0$.

a necessary condition for a constrained-efficient leverage:

$$\frac{1}{c_0} - \frac{\beta R}{c_1} = -\frac{f(\hat{z}^F(L_1|p_1))}{F(\hat{z}^F(L_1|p_1))}\frac{\partial \hat{z}^F(L_1|p_1)}{\partial L_1}\frac{L_1}{c_0}. \tag{28}$$

This condition is similar to the competitive equilibrium condition (24), once we set $\lambda = 0$. The reader may have expected to see terms reflecting the composite effect of leverage on the price together with the effect of the price on default thresholds appearing in the optimality condition of the planner (that is, chain-rule terms containing derivatives with respect to the price p_1 and of the price p_1 with respect to L_1). However, those first-order effects are equal to zero. The reason is that the first order effect on the fundamental default threshold of leverage L_1 operating through the price p_1, is zero. To see this, consider first the case $L_1 \in (\hat{L}, \overline{L})$. Noting that this implies $p_1 < z/R$ and using (15), we have that

$$\frac{\partial z^F(L_1|p_1)}{\partial p_1} = \hat{z}^F(L_1|p_1)\left[\frac{1+\beta}{z + p_1 - RL_1} - \frac{\beta R}{Rp_1 - \phi}\right] \tag{29}$$

which equals zero when evaluated at $p_1 = \mathcal{P}_1(L_1)$, using (25). For the case $L_1 < \hat{L}$, we have that $\mathcal{P}_1'(L_1) = 0$. Taken together, it thus follows that for any $L_1 < \overline{L}$, we have

$$\frac{\partial z^F(L_1|p_1)}{\partial p_1}\mathcal{P}_1'(L_1) = 0.$$

Hence there is no first-order effect of leverage on the default threshold operating through the price, p_1. A similar argument shows that the first-order effect of p_1 on the repayment value $V_1^R(n_1|p_1)$ is zero.

The reasons for these results is that, in an equilibrium, repaying banks are neither net sellers nor net buyers of capital (as they have to hold the capital stock). And thus, a change in price of capital does not have a first-order impact on their value.[15] This suggests that the equilibrium level of leverage is constrained efficient, a result we show next.

The following lemma will be helpful when comparing the planner's problem solution with a competitive equilibrium.

Lemma 7. *Suppose $\lambda = 0$. Let $p_1 = \mathcal{P}_1(L_1)$ for $L_1 < \overline{L}$. For any $\hat{p}_1 \in (\phi/R, z/R]$, we have:*

(i) $V_1^R((z + p_1)K - RKL_1|p_1) \leq V_1^R((z + \hat{p}_1)K - RKL_1|\hat{p}_1)$, *and*

[15] This also explains why the objective function is differentiable even though the function $P_1(L_1)$ is not differentiable at $L_1 = \hat{L}$.

(ii) $q_0(L_1|p_1) \leq q_0(L_1|\hat{p}_1)$,

with the first inequality is strict if $\hat{p}_1 \neq p_1$.

Proof. First, note that if we use $p > \phi/R$, it follows that

$$(z+p) - RL_1 > (z+\phi/R) - RL_1 = R\,(z+\phi/R)/R - L_1) = R(\overline{L} - L_1),$$

where we have used the definition of $\overline{L}$. Hence, for any $p \in (\phi/R, z/R]$ and $L_1 < \overline{L}$, $(z+p) - RL_1 > 0$.

Using Lemma 1, and that $V_1^R((z+p)K - RKL_1|p)$ is continuous in $p \in (\phi/R, z/R]$, we have that

$$V_1^R((z+p)K - RKL_1|p) = \hat{A} + (1+\beta)\log((z+p) - RL_1) + \beta \log\left(\frac{z-\phi}{p-\frac{\phi}{R}}\right) \equiv \hat{v}(p),$$

where $\hat{A} = A + (1+\beta)\log K$.

The function $v(p)$ is differentiable for $\phi/R < p$. And $v'(\bar{p}) = 0$ at $\bar{p} \equiv \beta z + (1+\beta)\phi/R - \beta RL_1 = \phi/R + \beta R(\overline{L} - L_1) > \phi/R$.

Note also that

$$\hat{v}''(\bar{p}) = \frac{1}{\beta(1+\beta)(R(\overline{L} - L_1))^2} > 0,$$

It follows that the function $\hat{v}(p)$ achieves a strict local minimum at $\bar{p}$. Given that $\hat{v}'(p)$ has a unique zero, it follows that the function $\hat{v}(p)$ is strictly decreasing in $p \in (\phi/R, \bar{p})$ and strictly increasing in $p > \bar{p}$.

Now, for $L_1 \in (\hat{L}, \overline{L})$, we have that $p_1 = \bar{p} \in (\phi/R, z/R)$. Then the result in part (i) is immediate, as $\hat{v}(p)$ has a strict minimum at $p = \bar{p} = p_1$. That is, for any $\hat{p}_1 \in (\phi/R, z/R]$, $\hat{v}(\hat{p}_1) \geq \hat{v}(p_1)$ with strict inequality if $\hat{p}_1 \neq p_1$.

For $L_1 \leq \hat{L}$, we have $p_1 = z/R$ and $\bar{p} \geq z/R$. Then, the result in part (i) is immediate, as $\hat{v}(p)$ is strictly decreasing in $p \in (\phi/R, z/R]$, and thus for any $\hat{p}_1 \in (\phi/R, z/R]$, $\hat{v}(\hat{p}_1) \geq \hat{v}(p_1)$ with strict inequality if $\hat{p}_1 \neq p_1$.

Given that V_1^R encounters a minimum at p_1, it follows then that the fundamental default threshold, $\hat{z}^F$, also encounters a minimum at p_1, as can be seen from (29). Part (ii) then follows. That is, the bond price q_0 also encounters a minimum at p_1, although it does so weakly, given that F could be constant in the relevant range.

□

The underlying economics at work in this lemma is that, in equilibrium, repaying banks are neither net buyers nor net sellers of capital in period 1. General equilibrium requires that banks hold the capital stock that remains after the default decisions have been made. If the price of capital were to deviate from its equilibrium value, a bank that repays in period 1 is able to choose the same allocation as in equilibrium. However, such a repaying bank could do strictly better by buying capital (when its price decreases) or selling capital (when its price increases), and thus it can do strictly better if the price deviates from the equilibrium one.

With this result at hand, we can prove a first-welfare theorem like result for our economy.

Proposition 1 (Constrained-efficiency)**.** *Suppose $\lambda = 0$. Any competitive equilibrium is constrained efficient.*

Proof. Consider an equilibrium. Let L^E denote the equilibrium level of leverage, and let $p_1^E = \mathcal{P}_1(L^E)$ denote the equilibrium price of capital in period 1.

Let L^P denote a level of leverage that solves the planner's problem (27). And let $p_1^P = \mathcal{P}_1(L^P)$ denote the price of capital in period 1 that corresponds to L^P.

Let $\mathbb{E}V_1(l_1, k_1; p_1)$ denotes the expected value for the bank in period 1. Using $\lambda = 0$,

$$\mathbb{E}V_1(l_1, k_1|p_1) \equiv \int_{\underline{z}}^{\overline{z}} \max\{V_1^R((z+p_1)K - Rl_1K|p_1), V_1^D(K, \tilde{z})\} dF(\tilde{z}),$$

where we have used that $\hat{z}^F$ is the point of indifference.

Constrained efficiency requires that

$$u(zK - RB_0 + q_0(L^P|p_1^P)L^PK) + \beta\mathbb{E}V_1(L^P, K|p_1^P) \geq u(zK - RB_0 + q_0(L^E, p_1^E)L^EK) + \beta\mathbb{E}V_1(L^E, K|p_1^E), \quad (30)$$

given that the planner could always have chosen the equilibrium level of leverage.

In the competitive equilibrium, banks at time $t = 0$ prefer to borrow $B_1 = L^EK$ rather than L^PK when facing equilibrium prices p_1^E. That is, it must be that

$$u(zK - RB_0 + q_0(L^E|p_1^E)L^EK) + \beta\mathbb{E}V_1(L^E, K|p_1^E) \geq u(zK - RB_0 + q_0(L^P|p_1^E)L^PK) + \beta\mathbb{E}V_1(L^P, K|p_1^E). \quad (31)$$

Part (ii) of Lemma 7 implies that $q_0(L^P|p_1^E) \geq q_0(L^P|p_1^P)$. For $L^P \leq 0$, Lemma 2 implies that $q_0(L^P|p_1^P) = q_0(L^P|p_1^E) = 1$. Thus, it follows that

$$q_0(L^P|p_1^E)L^PK \geq q_0(L^P|p_1^P)L^PK.$$

And hence,

$$zK - RB_0 + q_0(L^P|p_1^E)L^PK \geq zK - RB_0 + q_0(L^P|p_1^P)L^PK.$$

Part (i) of Lemma 7 also implies that $\mathbb{E}V_1(L^P, K|p_1^E) \geq \mathbb{E}V_1(L^P, K|p_1^P)$. Taken together with (31), we have that

$$u(zK - RB_0 + q_0(L^E|p_1^E)L^EK) + \beta\mathbb{E}V_1(L^E, K|p_1^E) \geq u(zK - RB_0 + q_0(L^P|p_1^P)L^PK) + \beta\mathbb{E}V_1(L^P, K|p_1^P) \quad (32)$$

Taken together, (30) and (32) imply that the competitive equilibrium choice of leverage also solves the planning problem. □

We conclude this section by highlighting a final result that must be satisfied in an equilibrium. Given that a competitive equilibrium is constrained efficient, it must satisfy the first-order condition (28). From the functional forms for $\hat{z}^F$ and V_1^R and that $q(L_1|p_1) = F(\hat{z}^F(L_1|p_1))$, it follows that

$$\frac{z+p_1}{p_0} - R = \frac{(1+\beta)RL_1}{p_0}\left(\frac{f(\hat{z}^F(L_1|p_1))\hat{z}^F(L_1|p_1)}{F(\hat{z}^F(L_1|p_1))} + \frac{1-F(\hat{z}^F(L_1|p_1))}{1+\beta}\right) \geq 0,$$

where $p_1 = \mathcal{P}_1(L_1)$ and $p_0 = \mathcal{P}_0(L_1)$. That is, the return to capital from period $t = 0$ to $t = 1$ must be weakly higher than the return on bonds, confirming a necessary condition for banks to hold $K > 0$ units of capital at the end of period $t = 0$.

4.2 Uniqueness and Existence of Competitive Equilibria Without Runs

In the previous section, we showed that the competitive equilibrium, if it exists, is constrained efficient. That is, the equilibrium leverage choice must solve the planning problem (27). Given that the leverage choice uniquely determines the rest of the equilibrium allocation, this immediately implies that if the planner problem has a unique solution, then the competitive equilibrium, if it exists, is also unique. This already implies that, generically, the competitive equilibrium is unique. In the following proposition, we formalize and strengthen this result.

Proposition 2 (Uniqueness)**.** *Suppose $\lambda = 0$ and that either*

(i) there is a unique solution to the planner problem (27)*, or*

(ii) there exists a competitive equilibrium with leverage $L_1 = B_1/K > \hat{L}$.

Then, there is at most one (symmetric pure-strategy) competitive equilibrium.

Proof. For part (i). This follows immediately from the fact that a competitive equilibrium is constrained efficient (Proposition 1) and that there is a unique leverage choice that solves the planning problem.

For part (ii). Suppose we have two distinct equilibria with elements indexed by A and B. Let $L^A < \overline{L}$ and $L^B < \overline{L}$ be the associated levels of leverage. Given that equilibria are characterized by the level of leverage, L^A and L^B must be different. Without loss of generality, we let $L^B < L^A$.

Given that $L^A > \hat{L}$, equation (25) implies that $p_1^A \neq p_1^B$. Facing prices for capital (p_0^A, p_1^A) banks must be willing to borrow $B_1^A = L^A K$ and invest K in period $t = 0$. This must be preferred to borrowing $B_1^B = L^B K$ and investing K in period $t = 0$ given prices (p_0^A, p_1^A). That is,

$$u(zK - RB_0 + q_0(L^A|p_1^A)L^A K) + \beta\mathbb{E}V_1(L^A, K|p_1^A) \geq u(zK - RB_0 + q_0(L^B|p_1^A)L^B K) + \beta\mathbb{E}V_1(L^B, K|p_1^A)$$

where we have used the definition of $\mathbb{E}V_1$ in the proof of Proposition 1. Lemma 7 implies that $q_0(L^B|p_1^A) \geq$

$q_0(L^A|p_1^B)$. Part (i) of the lemma then implies that $\mathbb{E}V_1(L^B, K|p_1^A) > \mathbb{E}V_1(L^B, K|p_1^B)$, given that $p_1^B \neq p_1^A$ and that default does not occur with probability one in any equilibrium (Lemma 3). Thus, it follows that

$$u(zK - RB_0 + q_0(L^B|p_1^A)L^BK) + \beta\mathbb{E}V_1(L^B, K|p_1^A) > u(zK - RB_0 + q_0(L^B, p_1^B)L^BK) + \beta\mathbb{E}V_1(L^B, K|p_1^B)$$

Taken together, the above implies that the time 0 utility to banks in the competitive equilibrium A is strictly higher than the one in the competitive equilibrium B. A contradiction of Proposition 1 as we found a competitive equilibrium that delivers strictly lower welfare than the constrained-efficient solution. □

As we argued above, there exists a solution to the planner's problem as long as the constraint set is non-empty. So let us consider a planner's solution with leverage L_1. Given the associated capital prices, for a competitive equilibrium to exist, we require that the optimal bank's choice of leverage at time $t = 0$ coincide with $L1$. The payoff function of the bank problem equals that of the planner at L_1. And in addition, both the planner's and the bank's payoffs have a zero derivative at L_1 (a result that we show below under Assumption 1). However, this is not enough to guarantee existence. A second-welfare theorem like result requires additional restrictions to ensure that the bank's problem has a global maximum at L_1. We will impose the following (sufficient) condition on the density for the outside option shock f.

Assumption 3 (Density)**.** *The probability distribution function f is such that $\left[\frac{1-F(z)}{1+\beta} + \frac{f(z)}{F(z)}z\right]$ is decreasing in z for any $z \in [\underline{z}, \overline{z}]$.*

With this, we can show the following result.

Proposition 3 (Existence of competitive equilibrium)**.** *Suppose $\lambda = 0$ and that Assumptions 1, 2, and 3 hold. If, in addition,*

$$\mathcal{P}_0(L_1) > \sup_{l_1 < (z+p_1)/R} \{q_0(l_1|\mathcal{P}_1(L_1))l_1\},$$

where L_1 solves the planning problem, then there exists a competitive equilibrium.

Proof. See Appendix A.1 □

In the proof, we show that Assumption 2, together with the Laffer curve condition in the Proposition, implies that the derivative of the bank's objective in (21) with respect to leverage can switch signs only once (from a positive value to a negative one). This derivative is zero at the planner's solution L_1, and thus $l_1 = L_1$ is a global optimizer for the bank's problem. Note that the Laffer curve condition is necessary for an equilibrium to exist, given Lemma 5.

To summarize, in this section we studied the efficiency, existence, and uniquenesss of competitive equilibrium in the absence of runs (i.e., $\lambda = 0$). We showed that the competitive equilibrium is constrained efficient. That is, the banks' private optimal leverage choice coincides with that which would have been chosen by a benevolent planner. As we show next, this result no longer holds in the presence of runs.

5 Optimal Regulation with Bank Runs

We now examine the constrained-efficient leverage in the environment with runs, $\lambda > 0$. We will show that the competitive equilibrium may no longer be constrained efficient.

Taking into account that $\lambda > 0$, the planner's problem now becomes:

$$\max_{L_1, c_0} \left\{ u(c_0) + \beta \int_{\underline{z}}^{\overline{z}} \left[d_1(L_1, \tilde{z}|p_1) V_1^D(K, \tilde{z}) + (1 - d_1(L_1, \tilde{z}|p_1)) V_1^R(n_1|p_1) \right] dF(\tilde{z}) \right\}, \tag{33}$$

subject to:

$$0 \leq c_0 \leq zK - RB_0 + q_0(L_1|p_1)L_1K,$$

$$L_1 < \overline{L},$$

and where:

$n_1 = (z + p_1)K - RL_1K, \ p_1 = \mathcal{P}_1(L_1)$, and d_1 is given by (18), as defined in Section 3.1.

Note that the planner's utility would be highest if λ were 0. That is, the planner would prefer that banks operate in an environment without runs. To see this, note that p_1 is independent of λ given L_1. For the same L_1, the planner attains a higher revenue per bond issuance in period $t = 0$, $q_0(L_1|p_1)L_1$, in the model without runs ($\lambda = 0$) than in the model with runs ($\lambda > 0$). Given that $V_1^R(n_1|p_1) \geq V_1^D(K, \tilde{z})$, for all $\tilde{z}$ where $d_1(L_1, \tilde{z}|p_1) \neq 0$, the planner prefers that banks operate in an environment without runs.

If in a competitive equilibrium with runs, the price of capital is such that $p_1 = z/R$, then the presence of runs does not impact the equilibrium, as $\hat{z}^F = \hat{z}^{Run}$ (see the discussion around condition (17)). The competitive equilibrium remains constrained efficient.

Let us focus now on the case where in equilibrium, $p_1 < z/R$, or equivalently, $L_1 > \hat{L}$. Assumption 1 guarantees the differentiability of the objective in the constrained-efficient problem

problem. And the first-order optimality condition yields the following:[16]

$$
\begin{aligned}
\frac{1}{c_0} - \frac{\beta R}{c_1} = & -\frac{(1-\lambda) f(\hat{z}^F(L_1|p_1)) \frac{\partial \hat{z}^F(L_1|p_1)}{\partial L_1} + \lambda f(\hat{z}^{Run}(L_1|p_1)) \frac{\partial \hat{z}^{Run}(L_1|p_1)}{\partial L_1}}{q_0(L_1|p_1)} \frac{L_1}{c_0} \\
& - \frac{\lambda f(\hat{z}^{Run}(L_1|p_1)) \frac{\partial \hat{z}^{Run}(L_1|p_1)}{\partial L_1}}{q_0(L_1|p_1)} \frac{\beta}{K} \left[V_1^R(n_1|p_1) - V_1^D(K, \hat{z}^{Run}(L_1|p_1))) \right] \\
& - \frac{\lambda f(\hat{z}^{Run}(L_1|p_1))}{q_0(L_1|p_1)} \underbrace{\frac{\partial \hat{z}^{Run}(L_1^*|p_1^*)}{\partial p_1} \mathcal{P}_1'(L_1)}_{\text{G.E.}} \left[\frac{L_1}{c_0} + \frac{\beta}{K} \left[V_1^R(n_1|p_1) - V_1^D(K, \hat{z}^{Run}(L_1|p_1)) \right] \right] \quad (34)
\end{aligned}
$$

for $L_1 \in (\hat{L}, \overline{L})$ and where $c_0 = zK - RB_0 - q_0(L_1|p1)L_1K$, $c_1 = \frac{1}{1+\beta}(z + p_1 - RL_1)K$, $n_1 = (z+p_1)K - RL_1K$, and $p_1 = \mathcal{P}_1(L_1)$.

The first two lines in this equation are are analogous to the optimality condition for individual banks (24). That is, the planner internalizes that higher borrowing reduces the bond price at which it can issue bonds. The third line encodes the general equilibrium effect by which the planner perceives a higher marginal cost of borrowing than the banks. That is, the constrained-efficient planner internalizes that an increase in aggregate leverage lowers the price of capital in period 1, and this lowers the run threshold, as reflected in the term $\frac{\partial \hat{z}^{Run}}{\partial p_1} \mathcal{P}_1'(L_1)$. The resulting increase in the probability of facing a run induces a lower price of bond issuances (the term multiplying L_1/c_0) and a higher expected loss due to runs (the term multiplying $\frac{\beta}{K}(V_1^R - V_1^D)$). These costs are not internalized by individual banks, potentially leading to more leverage in the competitive equilibrium than what is efficient.

To see more clearly that the inefficiency of the competitive equilibrium induces excessive leverage, we differentiate (16) and obtain

$$
\frac{\partial \hat{z}^{Run}(L_1|p_1)}{\partial p_1} = \hat{z}^{Run}(L_1|p_1) \frac{(1+\beta)\phi}{R(z + p_1 - RL_1)p_1} > 0, \quad (35)
$$

A decrease in the price of capital raises the default threshold because a bank facing a run is a net seller of capital. That is, when the bank cannot roll over the deposits, it is forced to sell capital to repay the deposits that are being withdrawn. In this context, *a decrease in the price of capital hurts the bank that is facing the run, making default more attractive.*[17]

[16] See Appendix A.3 for the derivation.

[17] This distinct impact of the price of capital on banks facing a run versus those at the fundamental threshold was a key feature that we highlighted in Amador and Bianchi (2024) in the context of a credit easing policy.

In addition, using (25) we also have that

$$\mathcal{P}_1'(L_1) = -\beta R < 0 \tag{36}$$

for $L_1 \in (\hat{L}, \overline{L})$. To the extent that banks are borrowing constrained in period 1, an increase in leverage implies a decrease in period 1 asset prices.

Putting together (35) and (36) confirms that the planner perceives a higher cost of leverage relative to individual banks in the competitive equilibrium. In particular, the planner internalizes that a marginal increase in leverage lowers asset prices in period 1 and lowers the default threshold in the presence of runs. That is, when banks choose their leverage in period 0, they do not internalize that higher leverage would contribute to lower asset prices and make other banks more vulnerable to runs.[18]

We summarize these results in the following proposition.

Proposition 4 (Constrained inefficiency in the presence of runs)**.** *Consider the case $\lambda > 0$. Any competitive equilibrium with leverage L_1 such that $z/\mathcal{P}_1(L_1) > R$ and where $f(\hat{z}^{Run}(L_1|\mathcal{P}_1(L_1)) > 0$ is constrained inefficient.*

Proof. This follows directly from noticing that the necessary conditions for the equilibrium choice of leverage and for the constrained-efficient level of leverage are different. □

We can flesh out the direction of the inefficiency by measuring the tax that would be necessary to reconcile the first-order conditions of the constrained planner with the equilibrium.

To see this, consider introducing a linear tax on banks' borrowing rebated lump-sum in period $t = 0$. In this case, the banks' budget constraint in period $t = 0$ becomes:

$$c_0 = (z + p_0)k_0 - Rb_0 + \frac{q_0(b_1, k_1)}{1 + \tau} b_1 - p_0 k_1 + T_0, \tag{37}$$

where τ_0 denotes the tax rate (or wedge) on borrowing, and T_0 the lump-sum transfer. In equilibrium, the government budget constraint implies $T_0 = \frac{\tau_0}{1+\tau_0} q_0(L_1)L_1 K$.

[18] It is worth highlighting that in general, a change in L_1 may affect both thresholds through changes in p_1. In the present model, however, there are no effects on $\hat{z}^F$ because, in the absence of runs, banks are neither net sellers nor net buyers of capital and, thus, are unaffected by a marginal change in asset prices. Notice that it is possible to modify the framework so that changes in L_1 also affect the fundamental default threshold. However, even in that scenario, the last term in (34) would vanish. Thus, highlighting the crucial role of bank runs in determining the need for banking regulation.

Implementing the constrained-efficient leverage requires setting the tax such that

$$\tau_0^* = -\frac{\partial \hat{z}^{Run}(L_1^*|p_1^*)}{\partial p_1}\mathcal{P}_1'(L_1^*)\left[\frac{\lambda f(\hat{z}^{Run}(L_1^*|p_1^*))}{(1-\lambda)F(\hat{z}^F(L_1^*|p_1^*)) + \lambda F(\hat{z}^{Run}(L_1^*|p_1^*))}\right]\frac{c_1^*}{\beta R} \times\left[\frac{L_1^*}{c_0^*} + \frac{\beta}{K}\Big(V_1^R(n_1^*|p_1^*) - V_1^D(K, \hat{z}^{Run}(L_1^*|p_1^*))\Big)\right], \quad (38)$$

where the superscript * denotes a constrained-efficient allocation. The expression corresponds to the wedge between the Euler equation for the constrained planner (34) and for individual banks (24) normalized by period 1 marginal utility. If $\lambda > 0$, and $f(\hat{z}^{Run}) > 0$, then $\tau_0^* > 0$. That is, implementing the constrained-efficient leverage requires a strictly positive tax on borrowing.

Numerical illustration. Figure 1 illustrates the difference in leverage between the competitive equilibrium and constrained-efficient solution for $\lambda \in [0.1]$. Panel (a) shows that at $\lambda = 0$, the competitive equilibrium and constrained-efficient leverage coincide, and as the probability of a run increases, leverage falls in both the competitive equilibrium and constrained-efficient solution. As a result of the externality uncovered above, the planner reduces leverage by more than individual banks as λ increases. That is, a higher probability of a run makes it more costly for banks to borrow, but the planner perceives an even higher marginal cost, and that is why it reduces leverage even more.

In line with this result, panel (b) shows that the equilibrium share of defaulting banks is higher in the competitive equilibrium. Notice that a higher probability of a run does not necessarily induce a higher share of defaulting banks. While for a given level of leverage, the share of defaulting banks increases with λ, as illustrated in panel (c), banks may choose to scale down leverage as a precautionary response to the higher likelihood of runs.

Decentralization. We now turn to the decentralization of the constrained-efficient allocations. Let's return to the linear tax introduced in (37). Looking at the tax formula in (38), we note that three elements determine the magnitude of the tax: how the run threshold varies with the asset price, how the price varies with leverage, and the losses that arise from a run. In particular, the tax is higher when the price of capital is more responsive to leverage, when the run threshold is more responsive to asset prices, and when there is a larger loss from runs.

An alternative policy that can implement the constrained-efficient leverage is a capital requirement constraint. In particular, suppose that at $t = 0$, banks must keep a ratio of net worth over assets higher than κ:

$$\frac{(z + p_1)k_1 - Rb_1}{p_0 k_1} \geq \kappa \quad (39)$$

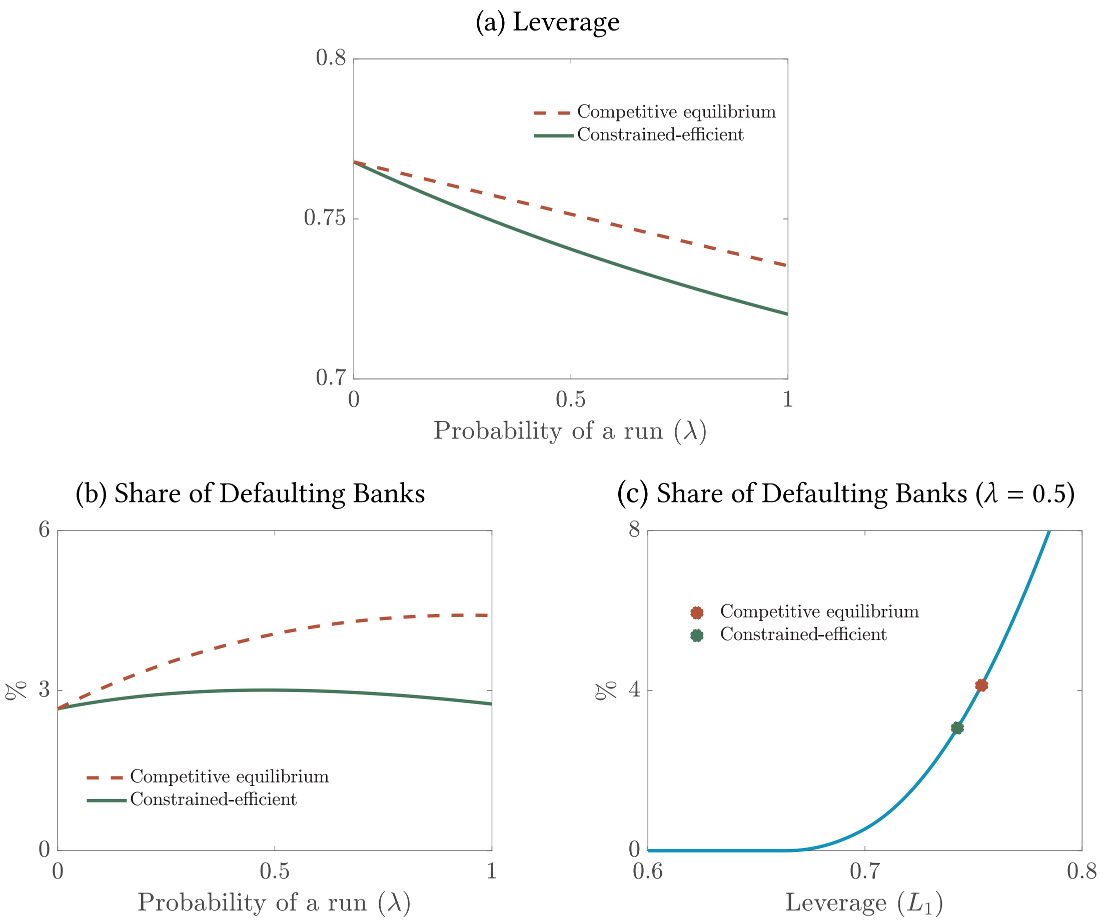

Figure 1: Simulations

Notes: Panels (a) and (b) present respectively leverage and the share of defaulting banks for the competitive equilibrium and constrained-efficient solution as a function of λ. Panel (c) sets $\lambda = 0.5$ and presents the share of defaulting banks for a range of leverage values. The solid dots represent the competitive equilibrium and constrained-efficient leverage for $\lambda = 0.5$. The simulation is generated using $\beta = 0.90$, $R = 1.04$, $z = 1$, $\phi = 0.2$, and $B_0 = 1.5$. The density for z_1^D is a triangular distribution with support [0,0.6].

By setting $\kappa = \frac{(z+p_1^*)-RL_1^*}{p_0^*}$ the central bank can implement the desired leverage. The idea is that when banks face prices as in the constrained-efficient solution, they would like to borrow more than the planner. A tight capital requirement constraint thus prevents banks from borrowing excessively.

Discussion. The analysis we have conducted rationalizes the type of macroprudential policies that have become a cornerstone of Basel III regulation. Under the macroprudential paradigm, regulations are adjusted throughout the cycle with the aim of containing systemic risk. For example, banks are required to maintain countercyclical capital buffers when vulnerabilities are building up.[19] We note that while both the leverage tax and the capital requirement implement the constrained-efficient leverage, they may have, in practice, different cyclical properties. We think that an important avenue for future research is a quantitative study of the cyclical properties of these policies.

We also note that we have focused exclusively on macroprudential policies and abstracted from other financial regulatory policies, such as deposit insurance or lender of last resort. While these are important policies in practice, we follow this route to isolate the new lessons that emerge from our analysis. For example, deposit insurance generally requires ex-ante restrictions in leverage because of moral hazard considerations. An analysis of how the scope for macroprudential policy highlighted here interacts with other government policies is an interesting area for future research.

6 Conclusions

We analyze the scope for banking regulation in a general equilibrium model where banks may default on their debt obligations because of fundamentals or self-fulfilling runs. We show that the competitive equilibrium is constrained efficient in the absence of runs. However, banks borrow too much relative to the social optimum in the presence of runs. The planner internalizes that lower leverage helps keep asset prices higher, making banks less vulnerable to self-fulfilling runs.

The aftermath of the banking turmoil of March 2023 has sparked renewed discussions on banking regulation. Echoing previous debates following the Great Financial Crisis of 2008, many proposals for tightening banking regulation are aimed at preventing bank bailouts. In our model, raising banks' capital requirements is desirable even in the absence of such bailouts. The basic message is that leverage and financial distress result in lower asset prices, and in turn, low asset prices make the economy more vulnerable to the damaging effects of self-fulfilling bank runs.

[19] Taxes on leverage have not received as much attention in policy circles, with some exceptions (e.g., the Minneapolis Plan to End Too Big To Fail). An alternative to a leverage tax, which is more commonly used, is a reserve requirement where banks must hold a fraction of deposits as unremunerated reserves at the central bank. A return on reserves lower than the rate on deposits implies that reserve requirements raise the cost of leverage.

References

Aguiar, Mark and Manuel Amador, "A contraction for sovereign debt models," *Journal of Economic Theory*, 2019, *183*, 842–875.

— , — , **Hugo Hopenhayn, and Ivan Werning**, "Take the short route: Equilibrium default and debt maturity," *Econometrica*, 2019, *87* (2), 423–462.

Allen, Franklin and Douglas Gale, "Financial contagion," *Journal of Political Economy*, 2000, *108* (1), 1–33.

Alvarez, Fernando and Urban Jermann, "Efficiency, equilibrium, and asset pricing with risk of default," *Econometrica*, 2000, *68* (4), 775–797.

Amador, Manuel and Javier Bianchi, "Bank runs, fragility, and credit easing," *Forthcoming, American Economic Review*, 2024.

Araujo, Aloisio, Mário Rui Páscoa, and Juan Pablo Torres-Martínez, "Collateral avoids Ponzi schemes in incomplete markets," *Econometrica*, 2002, *70* (4), 1613–1638.

Auclert, Adrien and Matthew Rognlie, "Unique equilibrium in the Eaton–Gersovitz model of sovereign debt," *Journal of Monetary Economics*, 2016, *84*, 134–146.

Bianchi, Javier, "Overborrowing and systemic externalities in the business cycle," *American Economic Review*, 2011, *101* (7), 3400–3426.

— **and Enrique G Mendoza**, "Optimal time-consistent macroprudential policy," *Journal of Political Economy*, 2018, *126* (2), 588–634.

Bloise, Gaetano and Pietro Reichlin, "Asset prices, debt constraints and inefficiency," *Journal of Economic Theory*, 2011, *146* (4), 1520–1546.

Bocola, Luigi and Alessandro Dovis, "Self-fulfilling debt crises: A quantitative analysis," *American Economic Review*, 2019, *109* (12), 4343–4377.

Calomiris, Charles W and Charles M Kahn, "The role of demandable debt in structuring optimal banking arrangements," *American Economic Review*, 1991, *81* (3), 497–513.

Chatterjee, Satyajit and Burcu Eyigungor, "Maturity, indebtedness, and default risk," *American Economic Review*, 2012, *102* (6), 2674–2699.

Cole, Harold L. and Timothy J. Kehoe, "Self-fulfilling debt crises," *Review of Economic Studies*, 2000, *67(1)*, 91–116.

Cooper, Russell and Thomas W Ross, "Bank runs: Liquidity costs and investment distortions," *Journal of Monetary Economics*, 1998, *41* (1), 27–38.

Dávila, Eduardo and Anton Korinek, "Pecuniary externalities in economies with financial frictions," *The Review of Economic Studies*, 2018, *85* (1), 352–395.

_ **and Itay Goldstein**, "Optimal deposit insurance," *Journal of Political Economy*, 2023, *131* (7), 1676–1730.

Diamond, Douglas W and Anil K Kashyap, "Liquidity requirements, liquidity choice, and financial stability," in "Handbook of Macroeconomics," Vol. 2, Elsevier, 2016, pp. 2263–2303.

_ **and Philip H Dybvig**, "Bank runs, deposit insurance, and liquidity," *Journal of Political Economy*, 1983, *91(3)*, 401–419.

_ **and Raghuram G Rajan**, "A theory of bank capital," *the Journal of Finance*, 2000, *55* (6), 2431–2465.

DiTella, Sebastian, "Optimal regulation of financial intermediaries," *American Economic Review*, 2019, *109* (1), 271–313.

Drechsler, Itamar, Alexi Savov, Philipp Schnabl, and Olivier Wang, "Banking on uninsured deposits," 2023. NBER Working Paper No. 31138.

Dubey, Pradeep, John Geanakoplos, and Martin Shubik, "Default and punishment in general equilibrium," *Econometrica*, 2005, *73* (1), 1–37.

Egan, Mark, Ali Hortaçsu, and Gregor Matvos, "Deposit competition and financial fragility: Evidence from the us banking sector," *American Economic Review*, 2017, *107* (1), 169–216.

Ennis, Huberto M and Todd Keister, "Bank runs and institutions: The perils of intervention," *American Economic Review*, 2009, *99* (4), 1588–1607.

Farhi, Emmanuel, Mikhail Golosov, and Aleh Tsyvinski, "A theory of liquidity and regulation of financial intermediation," *The Review of Economic Studies*, 2009, *76* (3), 973–992.

Geanakoplos, John and Heracles M Polemarchakis, "Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete," 1985. working paper.

Gersbach, Hans and Jean-Charles Rochet, "Aggregate investment externalities and macroprudential regulation," *Journal of Money, Credit and Banking*, 2012, *44*, 73–109.

Gertler, Mark and Nobuhiro Kiyotaki, "Banking, liquidity, and bank runs in an infinite horizon economy," *American Economic Review*, 2015, *105* (7), 2011–2043.

_ , _ , and Andrea Prestipino, "Credit booms, financial crises, and macroprudential policy," *Review of Economic Dynamics*, 2020, *37*, S8–S33.

Gromb, Denis and Dimitri Vayanos, "Equilibrium and welfare in markets with financially constrained arbitrageurs," *Journal of Financial Economics*, 2002, *66* (2-3), 361–407.

Hart, Oliver D, "On the optimality of equilibrium when the market structure is incomplete," *Journal of Economic theory*, 1975, *11* (3), 418–443.

He, Zhiguo and Péter Kondor, "Inefficient investment waves," *Econometrica*, 2016, *84* (2), 735–780.

Jeske, Karsten, "Private international debt with risk of repudiation," *Journal of Political Economy*, 2006, *114* (3), 576–593.

Jiang, Erica Xuewei, Gregor Matvos, Tomasz Piskorski, and Amit Seru, "Monetary tightening and U.S. bank fragility in 2023: Mark-to-market losses and uninsured depositor runs?," 2023. NBER Working Paper No. 31048.

Kashyap, Anil K, Dimitrios P Tsomocos, and Alexandros P Vardoulakis, "Optimal bank regulation in the presence of credit and run risk," *Journal of Political Economy*, 2024, *132* (3), 000–000.

Kehoe, Timothy J and David K Levine, "Debt-constrained asset markets," *The Review of Economic Studies*, 1993, *60* (4), 865–888.

Keister, Todd, "Bailouts and financial fragility," *Review of Economic Studies*, 2016, *83* (2), 704–736.

Kilenthong, Weerachart T and Robert M Townsend, "A market-based solution for fire sales and other pecuniary externalities," *Journal of Political Economy*, 2021, *129* (4), 981–1010.

Kurlat, Pablo, "Investment externalities in models of fire sales," *Journal of Monetary Economics*, 2021, *122*, 102–118.

Lanteri, Andrea and Adriano A Rampini, "Constrained-efficient capital reallocation," *American Economic Review*, 2023, *113* (2), 354–395.

Lorenzoni, Guido, "Inefficient Credit Booms," *Review of Economic Studies*, 2008, *75* (3), 809–833.

Martins-Da-Rocha, V Filipe, Toan Phan, and Yiannis Vailakis, "Pecuniary externalities in competitive economies with limited pledgeability," 2022. Federal Reserve Bank of Richmond.

Metrick, Andrew, "The Failure of Silicon Valley Bank and the Panic of 2023," *Journal of Economic Perspectives*, 2024, *38* (1), 133–152.

Ottonello, Pablo, Diego J Perez, and Paolo Varraso, "Are collateral-constraint models ready for macroprudential policy design?," *Journal of International Economics*, 2022, *139*, 103650.

Stein, Jeremy, "Monetary policy as financial stability regulation," *Quarterly Journal of Economics*, 2012, *127* (1), 57–95.

Stiglitz, Joseph E, "The inefficiency of the stock market equilibrium," *The Review of Economic Studies*, 1982, *49* (2), 241–261.

Tella, Sebastian Di and Pablo Kurlat, "Why are banks exposed to monetary policy?," *American Economic Journal: Macroeconomics*, 2021, *13* (4), 295–340.

Uhlig, Harald, "A model of a systemic bank run," *Journal of Monetary Economics*, 2010, *57* (1), 78–96.

Online Appendix to "Bank Runs, Fragility, and Regulation

Manuel Amador and Javier Bianchi

A Omitted Proofs

A.1 Proof of Proposition 3

Assumption 2 guarantees a solution to the planning problem with $n_0 > 0$. Let L_1 be a planner's solution and $p_0 = \mathcal{P}_0(L_0)$ and $p_1 = \mathcal{P}_1(L_1)$ be the associated capital prices. We are now going to show that $l_1 = L_1$ is a solution to the bank's problem given the prices.

We already know that $c_0 = c_0(n_0)$ and that $k_1 = k_1(n_0, l_1|p_0, p_1)$, as given by (22) and (23). The only remaining choice is l_1. We have already argue that a choice of leverage that leads to default for certain is not optimal (see proof of Lemma 3). This implies that $n_1 > 0$, given (15). Thus we can restrict the choice of l_1 to $l_1 < (z + p_1)/R$ or, more strictly, to values such that $F(\hat{z}^F(l_1|p_1)) > 0$.

We can then rewrite the objective function of the bank's problem as:

$$W(l_1|p_1) = \log(c_0(n_0)) + \beta F(\hat{z}^F(l_1|p_1))V_1^R\Big((z + p_1 - Rl_1)k_1\Big|p_1\Big) + \beta \int_{\hat{z}^F(l_1|p_1)}^{\infty} V_1^D(k_1, \tilde{z})dF(\tilde{z})$$

with $k_1 = k_1(n_0, l_1|p_0, p_1)$.

Using the functional forms, and that $p_1 \leq z/R$, we can rewrite the above as

$$W(l_1|p_1) = \log(c_0(n_0)) + \beta(1 + \beta)\log k_1 + \beta F(\hat{z}^F(l_1|p_1))V_1^R\Big((z + p_1 - Rl_1)\Big|p_1\Big) \\ + \beta \int_{\hat{z}^F(l_1|p_1)}^{\infty} V_1^D(1, \tilde{z})dF(\tilde{z})$$

Using the definition of $\hat{z}^F$ in (15), and that $p_1 \leq z/R$, we have that

$$\log(z^F(l_1|p_1)) = (1 + \beta)\log(z + p_1 - Rl_1) + \text{constant}(p_1),$$

where the constant term depends on the prices. Taking the derivative of $\hat{z}^F$ with respect to l_1, we have that

$$\frac{\partial \hat{z}^F(l_1|p_1)}{\partial l_1} = -\hat{z}^F(l_1|p_1)\frac{(1 + \beta)R}{z_1 + p_1 - Rl_1},$$

We can use this to write (eliminating the dependence on p_1 of the functions for simplicity):

$$W'(l_1|p_1) = \frac{\beta(1+\beta)^2 F(\hat{z}^F(l_1))}{(z+p_1-Rl_1)(p_0-F(\hat{z}^F(l_1))l_1)} \times \left[\frac{z+p_1-Rp_0}{1+\beta} - Rl_1\left(\frac{f(\hat{z}^F(l_1))\hat{z}^F(l_1)}{F(\hat{z}^F(l_1))} + \frac{1-F(\hat{z}^F(l_1))}{1+\beta}\right)\right],$$

where we used that $q(l_1) = F(\hat{z}^F(l_1))$ and that $p_0 - F(\hat{z}^F(l_1))l_1 > 0$ for all $l_1 < (z+p_1)/R$ by the assumption in the proposition. We also used that $F(\hat{z}^F(l_1)) > 0$, as it is not optimal for the bank to borrow into full default. Assumption 1 guarantees that W is indeed differentiable at the choice of l_1 so that $\hat{z}^F(l_1) = \overline{z}$.

Given that $\hat{z}^F(l_1)$ is weakly decreasing in l_1, Assumption 3 guarantees that

$$\frac{f(\hat{z}^F(l_1))\hat{z}^F(l_1)}{F(\hat{z}^F(l_1))} + \frac{(1-F(\hat{z}^F(l_1)))}{1+\beta}$$

is weakly increasing in leverage. Note also that this term is zero for any $l_1 \leq 0$ (as the bank is defaulting with probability 0 for $l_1 \leq 0$, Lemma 2). It follows that

$$Rl_1\left[\frac{f(\hat{z}^F(l_1))\hat{z}^F(l_1)}{F(\hat{z}^F(l_1))} + \frac{(1-F(\hat{z}^F(l_1)))}{1+\beta}\right],$$

The planner solution L_1 is such that $W'(L_1) = 0$, a result that follows from (28). So, $W'(l_1) \leq 0$ for $l_1 \leq L_1$ and $W'(l_1) \geq 0$ for $l_1 \geq L_1$, and thus the planner's solution L_1 is a global maximum of the bank's problem, confirming that last condition for the existence of an equilibrium. □

A.2 Derivation of Equation (24)

Let us define

$$\hat{V}_1^R(l_1) = A + (1+\beta)\log(z_1+p_1-Rl_1) + \beta\log\left(\frac{z-\phi}{p_1-\frac{\phi}{R_2}}\right) \tag{A.1}$$

$$\hat{V}_1^D(z_1^D) = \log(z_1^D) + \beta\log(z_2^D) \tag{A.2}$$

where we have removed the dependence on p_1 for simplicity (as it is taken as given in the banks' problem).

Using these two definitions and (2) and (10), we can first express (21) as

$$V_0(K,B_0) = \max_{k_1,l_1}\Bigg\{ \log((z+p_0)K + q_0(l_1)k_1l_1 - p_0k_1) + \beta(1+\beta)\log(k_1) + \beta F(\hat{z}^{Run}(l_1))\hat{V}_1^R(l_1) + \beta\int_{\hat{z}^F(l_1)}^{\infty} \hat{V}_1^D(z_1^D)dF(z_1^D) + \beta\int_{\hat{z}^{Run}(l_1)}^{\hat{z}^F(l_1)} \left[\lambda\hat{V}_1^D(z_1^D) + (1-\lambda)\hat{V}_1^R(l_1)\right] dF(z_1^D)\Bigg\},$$

Taking first-order condition with respect to l_1 yields

$$k_1\frac{q_0'(l_1)l_1 + q_0(l_1)}{c_0} + \beta f(\hat{z}^{Run}(l_1))\hat{V}_1^R(l_1)\frac{\partial\hat{z}^{Run}(l_1)}{\partial l_1} + \beta F(\hat{z}^{Run}(l_1))\frac{\partial\hat{V}_1^R(l_1)}{\partial l_1} - \beta f(\hat{z}^F(l_1))\hat{V}_1^D(\hat{z}^F(l_1))\frac{\partial\hat{z}^F(l_1)}{\partial l_1} + \beta f(\hat{z}^F(l_1))[\lambda\hat{V}_1^D(\hat{z}^F(l_1)) + (1-\lambda)\hat{V}_1^R(l_1)]\frac{\partial\hat{z}^F(l_1)}{\partial l_1} - \beta f(\hat{z}^{Run}(l_1))\left[\lambda\hat{V}_1^D(\hat{z}^{Run}(l_1)) + (1-\lambda)\hat{V}_1^R(l_1)\right]\frac{\partial\hat{z}^{Run}(l_1)}{\partial l_1} + \beta(1-\lambda)\int_{\hat{z}^{Run}(l_1)}^{\hat{z}^F(l_1)} \frac{\partial\hat{V}_1^R(l_1)}{\partial l_1}dF(z_1^D) = 0,$$

If we that $\hat{V}_1^R(l_1) = \hat{V}_1^D(\hat{z}^F(l_1))$, the expression above can be written as

$$k_1\frac{q_0'(l_1)l_1 + q_0(l_1)}{c_0} + \beta f(\hat{z}^{Run}(l_1))\lambda\left[\hat{V}_1^D(\hat{z}^{Run}(l_1)) - \hat{V}_1^R(l_1)\right]\frac{\partial\hat{z}^{Run}(l_1)}{\partial l_1} + \beta\left[(1-\lambda)F(\hat{z}^F(l_1)) + \lambda F(\hat{z}^{Run}(l_1))\right]\frac{\partial\hat{V}_1^R(l_1)}{\partial l_1} = 0 \quad \text{(A.3)}$$

In addition, using the definition of $\hat{V}^R$ and equation (19), we get

$$\frac{\partial\hat{V}_1^R(l_1)}{\partial l_1} = -\frac{(1+\beta)R}{n_1}k_1$$
$$q_0'(l_1) = (1-\lambda)f(\hat{z}^F(l_1))\frac{\partial\hat{z}^F(l_1)}{\partial l_1} + \lambda f(\hat{z}^{Run}(l_1))\frac{\partial\hat{z}^{Run}(l_1)}{\partial l_1},$$

Replacing these three conditions in (A.3), we obtain

$$\frac{k_1}{c_0}\left[\frac{\left[(1-\lambda)f(\hat{z}^F(l_1))\frac{\partial\hat{z}^F(l_1)}{\partial l_1} + \lambda f(\hat{z}^{Run}(l_1))\frac{\partial\hat{z}^{Run}(l_1)}{\partial l_1}\right]l_1}{(1-\lambda)F(\hat{z}^F(l_1)) + \lambda F(\hat{z}^{Run}(l_1))} + 1\right] - \beta\frac{\lambda f(\hat{z}^{Run}(l_1))\frac{\partial\hat{z}^{Run}(l_1)}{\partial l_1}}{(1-\lambda)F(\hat{z}^F(l_1)) + \lambda F(\hat{z}^{Run}(l_1))}\left[\hat{V}_1^D(\hat{z}^{Run}(l_1)) - \hat{V}_1^R(l_1)\right] = \beta\frac{(1+\beta)R}{n_1}k_1, \quad \text{(A.4)}$$

Finally, replacing c_1 from (11), using (A.1) and (A.2), and rearranging, we reach the desired expression, (24).

A.3 Derivation of Equation (34)

Let us define

$$\tilde{V}^R(L_1,p_1) = A + (1+\beta)\log(K) + (1+\beta)\log\left((1+\beta)\left(z_1 - RL_1 + \frac{\phi}{R}\right)K\right) + \beta\log\left(\frac{z-\phi}{p_1 - \frac{\phi}{R}}\right). \quad \text{(A.5)}$$

The argument in the text shows that

$$\frac{\partial \tilde{V}^R(L_1,\mathcal{P}_1(L_1))}{\partial p_1} = 0.$$

We can express the planning problem (27) as

$$\begin{aligned}\max_{L_1}\Bigg\{ &\log(zK - RB_0 + q_0(L_1|\mathcal{P}_1(L_1))KL_1) + \beta(1+\beta)\log(K) \\ &+ \beta F(\hat{z}^{Run}(L_1|\mathcal{P}(L_1)))\tilde{V}^R(L_1,\mathcal{P}_1(L_1)) \\ &+ \beta\Big[F(\hat{z}^F(L_1|\mathcal{P}(L_1))) - F(\hat{z}^{Run}(L_1|\mathcal{P}(L_1)))\Big](1-\lambda)\tilde{V}^R(L_1,\mathcal{P}_1(L_1)) \\ &+ \beta\int_{\hat{z}^{Run}(L_1|\mathcal{P}(L_1))}^{\hat{z}^F(l_1|\mathcal{P}(L_1))} \lambda V_1^D(z_1^D)dF(z_1^D) + \beta\int_{\hat{z}^F(L_1|\mathcal{P}(L_1))}^{\infty} V_1^D(z_1^D)dF(z_1^D)\Bigg\}, \quad \text{(A.6)}\end{aligned}$$

The first-order condition with respect to L_1 yields

$$\begin{aligned}&\frac{1}{c_0}\left[q_0(L_1|\mathcal{P}(L_1)) + \left(\frac{\partial q_0(L_1|\mathcal{P}(L_1))}{\partial L_1} + \frac{\partial q_0(L_1|\mathcal{P}(L_1))}{\partial p_1}\frac{\partial \mathcal{P}_1}{\partial L_1}\right)L_1\right]K \\ &+ \beta f(\hat{z}^{Run})\lambda(\tilde{V}_1^R(L_1) - V_1^D(\hat{z}^{Run}))\left[\frac{\partial \hat{z}^{Run}}{\partial L_1} + \frac{\partial \hat{z}^{Run}}{\partial p_1}\frac{\partial \mathcal{P}_1}{\partial L_1}\right] \\ &+ \beta[\lambda F(\hat{z}^{Run}) + (1-\lambda)F(\hat{z}^{Run})]\frac{\partial \tilde{V}_1^R}{\partial L_1} = 0, \quad \text{(A.7)}\end{aligned}$$

We also have

$$\frac{\partial \tilde{V}_1^R(L_1,\mathcal{P}_1(L_1))}{\partial L_1} = -u'(c_1^R)R. \quad \text{(A.8)}$$

Using this expression and replacing $\frac{\partial q_0(L_1|\mathcal{P}(L_1))}{\partial L_1}$ and (19) in (A.7), we arrive at (34).