

MACROPRUDENTIAL POLICY WITH LEAKAGES*

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Abstract

The outreach of macroprudential policies is likely limited in practice by imperfect regulation enforcement, whether due to shadow banking, regulatory arbitrage, or other regulation circumvention schemes. We study how such concerns affect the design of optimal regulatory policy in a workhorse model in which pecuniary externalities call for macroprudential taxes on debt, but with the addition of a novel constraint that financial regulators lack the ability to enforce taxes on a subset of agents. While regulated agents reduce risk taking in response to debt taxes, unregulated agents react to the safer environment by taking on more risk. These leakages undermine the effectiveness of macroprudential taxes but do not necessarily call for weaker interventions. A quantitative analysis of the model suggests that aggregate welfare gains and reductions in the severity and frequency of financial crises remain, on average, largely unaffected by even significant leakages.

Keywords: Macroprudential policy, regulatory arbitrage, financial crises, limited regulation enforcement

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1 Introduction

Macroprudential regulation has emerged as a central element of the post-financial crisis dominant policy paradigm, in which the management of credit cycles has been elevated to the rank of first-order policy concerns. Central to this view is the notion that curbing credit booms reduces financial sector vulnerabilities to future reversals in financial conditions. The effectiveness of macroprudential regulation, however, is not being taken for granted and has been the subject of a growing empirical literature (see, e.g., [Cerutti, Claessens and Laeven, 2017](#)). A common policy concern is that such new regulation could be bypassed and have unintended consequences.¹

A number of important conceptual questions arise against this backdrop: Is the scope for macroprudential policy significantly altered by the possibility of leakages? Does macroprudential policy remain desirable in the presence of leakages? How is the optimal design of regulation altered by the presence of leakages? The theoretical literature on macroprudential policy has so far abstracted from the practically relevant issue of imperfect regulation enforcement, and thus a proper framework to address these questions is lacking. Our goal in this paper is to fill this gap by providing a theoretical model suited to tackle these issues.

In line with the emerging policy paradigm and an established theoretical literature on the topic (e.g., [Lorenzoni 2008](#)), we adopt a framework in which inefficient private borrowing decisions generate excessive financial fragility. In our theory, a pecuniary externality resulting from frictions in financial markets makes macroprudential policy desirable, yet the introduction of such policy endogenously leads to increased risk taking by a shadow sector endowed with the ability to bypass regulation. These unintended spillover effects in turn feed into the economy's exposure to financial crises, limiting the effectiveness of macroprudential policy and altering its optimal design.

Our framework builds on [Bianchi \(2011\)](#)'s model of macroprudential policy, in which a pecuniary externality generates overborrowing and excessive exposure to financial crises. We extend this workhorse model by adding a shadow sector that is able to circumvent regulation. The macroprudential authority is able to curb risk taking within a narrow (regulated) sector, but has to internalize the (unregulated) shadow sector's destabilizing response to its actions.

We start our analysis with a tractable three-period model. In the model, agents initially make a borrowing decision and are then subject to income shocks in the intermediate period while facing a collateral constraint that limits their ability to smooth consumption. The presence of a market price in the collateral constraint implies that the higher the aggregate leverage chosen in the initial period, the larger the contraction in the borrowing capacity in the intermediate period for all households. This pecuniary externality and the associated financial amplification

¹ Empirical studies analyzing such leakages in macroprudential policies include [Magud et al. \(2011\)](#), [Aiyar, Calomiris and Wieladek \(2014\)](#), [Klein \(2012\)](#), [Jiménez, Ongena, Peydró and Saurina \(2012\)](#), [Forbes, Fratzscher and Straub \(2013\)](#), [Dassatti and Peydró \(2013\)](#), and [Ahnert, Forbes and Reinhardt \(2018\)](#).

effects are not internalized by private households, providing a rationale for a macroprudential policy aimed at limiting private leverage. Different from existing models, however, we assume that macroprudential regulation can only be enforced on a subset of the population. As a result, a macroprudential policy seeking to make regulated agents internalize the pecuniary externality creates a safer environment and has the unintended consequence of encouraging higher borrowing by unregulated agents. These spillovers undermine the effectiveness of macroprudential policy and increase the economy's exposure to financial crises.

We show that despite destabilizing spillover effects on the unregulated sphere, a small macroprudential tax on regulated agents is always strictly welfare improving for all agents. Going beyond a perturbation argument, we derive an explicit formula for the optimal macroprudential tax. Relative to the standard Pigouvian tax expression familiar from the literature, accounting for leakages yields two new elements. One reflects the weaker influence of the financial regulator onto the price of collateral due to the unregulated sector partially offsetting the regulated sector's response to the regulation. The second one reflects the regulator's higher valuation of relaxing future collateral constraints resulting from the unregulated sector's higher exposure to financial crises. Taken together, these two effects imply an ambiguous effect of leakages on the size of optimal macroprudential taxes.

We then turn to an infinite horizon model to pursue a quantitative analysis. In the infinite horizon context, leakages further give rise to a time inconsistency problem. To see why, suppose agents expect loose regulation in the future. This expectation generates a perception of a riskier environment, which induces unregulated agents to take less risk today, narrowing the gap between the borrowing choices of regulated and unregulated agents. Interestingly, this channel works despite unregulated agents not being directly subject to regulation. Given our goal of evaluating the extent to which leakages undermine the effectiveness of regulation, we find it useful to focus on a discretionary regime in which a regulator has limited ability to exploit the disciplining effect just described. We thus consider a government that sets optimal regulation sequentially and without commitment, and we focus on Markov-perfect equilibria. We then calibrate our infinite horizon model to match the features of emerging market crises and experiment with degrees of leakages ranging from 0 to 50% of agents making up the unregulated sphere of the economy.

The main take away from our quantitative analysis is that even subject to significant leakages, macroprudential policy remains highly effective and desirable. To offset the increase in borrowing by the unregulated sector, the planner induces even tighter regulation on the regulated sector, and this results in aggregate levels of borrowing that are comparable to the constrained-efficient allocation. Average welfare gains are also surprisingly stable with respect to leakages. Nonetheless, the welfare gains are spread unevenly in the economy, with the lion's share going to unregulated agents. Intuitively, while the financial stability benefits of macroprudential policy are partially

offset by the burden of regulation for regulated agents, they are not for unregulated agents, who free-ride on others' precautionary behavior.

This paper belongs to a growing literature providing foundations for macroprudential policies. A first strand of this literature, most closely related to our paper, examines pecuniary externalities and incomplete markets when financial constraints depend on market prices (Caballero and Krishnamurthy, 2001, Lorenzoni, 2008, Bianchi, 2011, Jeanne and Korinek, 2012, Bianchi and Mendoza, 2018, Korinek, 2018). Abstracting from imperfect regulation enforcement, this literature has tackled several issues, including the interactions between macroprudential and stabilization policies (Benigno et al., 2013), monetary policy (Fornaro, 2015, Ottonello, 2015, Sergeyev, 2016, Coulibaly, 2018, Devereux, Young and Yu, 2018), international coordination (Bengui, 2013), policy cyclicalities (Schmitt-Grohé and Uribe, 2017), multiple equilibria (Schmitt-Grohé and Uribe, 2016), news shocks (Bianchi, Liu and Mendoza, 2016), learning about financial innovation (Bianchi, Boz and Mendoza, 2012), volatility (Reyes-Heroles and Tenorio, 2018), trend shocks (Flemming et al., 2016, Seoane and Yurdagul, 2017), and fiscal capacity (Stavrakeva, 2017). A second strand of the literature on macroprudential policy examines aggregate demand externalities in the presence of nominal rigidities (Schmitt-Grohé and Uribe, 2013, Farhi and Werning, 2016). Recent research in this area has studied Mundell's Trilemma (Farhi and Werning, 2012), liquidity traps (Korinek and Simsek, 2016, Acharya and Bengui, 2018, Fornaro and Romei, 2018), and fiscal unions (Farhi and Werning, 2017).

A common theme in this literature is the presence of a wedge between the private and social cost of borrowing, which generates scope for Pigouvian taxes. Our contribution is to explicitly model the imperfect enforcement of macroprudential regulation, pursuing both a qualitative and quantitative analysis of optimal policy. The paper also speaks to an empirical literature that examines the effectiveness of macroprudential regulation.² In particular, our quantitative predictions are consistent with the findings that tighter regulation leads to opposite risk-taking responses by different agents, although tighter regulation leads to an overall reduction in risk taking (Dassatti and Peydró, 2013, Aiyar, Calomiris and Wieladek, 2014).

Finally, our paper relates to a literature in financial intermediation studying the interaction between regulation and shadow banking activity. Examples include Huang (2014), Plantin (2015), Grochulski and Zhang (2015), Farhi and Tirole (2017), Ordonez (2018), Ordonez and Piguillem (2018), Begenau and Landvoigt (2018) and Bengui, Bianchi and Coulibaly (2019). Begenau and Landvoigt, for example, study a model in which traditional banks are covered by deposit insurance but shadow banks are not, and find that a tightening of capital requirements for traditional banks causes shadow banks to take over a larger fraction of the credit market. We do not model banks explicitly but instead consider a generic setting with regulated and unregulated agents, and point to a distinct channel involving leverage responses to endogenous changes

²See footnote 1 for a list of papers.

in systemic risk generated by regulation. From a normative standpoint, the above mentioned literature adopts a primarily microprudential approach to financial regulation. In contrast, we provide a first analysis of *macroprudential* regulation in an economy with an unregulated sector.

The paper is organized as follows. Section 2 presents a three-period model that shows analytical results for the main mechanisms in the paper. Section 3 presents the results from the infinite horizon quantitative model. Section 4 concludes. Proofs are collected in the appendix.

2 A Three-Period Model

In this section, we present an analytically tractable three-period model of financial crises in which macroprudential policy can only be enforced on a subset of agents. Specific assumptions on preferences deliver closed-form solutions for continuation equilibria and help us obtain an analytical characterization of the optimal macroprudential policy problem in the presence of leakages. Later, in Section 3, we study the same mechanisms we outline in this simple model in the context of a more general quantitative infinite horizon model.

2.1 Environment

We consider a small open economy that lasts for three periods $t = 0, 1, 2$. There are two types of goods, tradables and nontradables, and no production. The only source of uncertainty is over the endowment of tradable goods in period $t = 1$.

The economy is populated by a continuum of agents of one of two types $i = U, R$ that differ in their ability to bypass regulation: a fraction γ is unregulated (U) and the remaining $1 - \gamma$ is regulated (R). The distinction between unregulated and regulated agents is intended to capture in a broad sense the existence of a shadow banking sector, differences in access to sources of funding, or differences in the ability to circumvent regulation by different agents.³

Both types of agents have identical preferences and endowments. Preferences are given by

$$W_i = c_{i0}^T + \mathbb{E}_0 [\beta \ln(c_{i1}) + \beta^2 \ln(c_{i2})] \quad (1)$$

with

$$c_{it} = (c_{it}^T)^\omega (c_{it}^N)^{1-\omega},$$

where \mathbb{E} is the expectation operator and $\beta < 1$ is a discount factor. Date 0 utility is linear in tradable consumption c^T , while date 1 and date 2 utility is logarithmic in a consumption basket c , which is a Cobb-Douglas aggregator of tradable goods consumption c^T and nontradable goods consumption c^N . The parameter ω is the share of tradables in total consumption.

³For the most part, we consider an environment where the size of the unregulated sector γ is exogenous, but will also provide a simple way to endogenize γ based on an idiosyncratic cost of bypassing regulation.

Agents receive endowments of tradable goods and nontradable goods of $(\{y_t^T(s)\}, y_t^N)$ at date 1 and date 2, but do not receive any endowment at date 0. The date 1 endowment of tradables $y_1^T(s)$ is a random variable depending on the event $s \in S$, which can be interpreted as the aggregate state of the economy. For simplicity, we assume that y_2^T is deterministic and that the nontradable goods endowment is constant, $y_1^N = y_2^N = y^N$.

Agents have access to a single, one-period, non-state-contingent bond denominated in units of tradable goods that pays a fixed interest rate r , determined exogenously in the world market. Denoting the relative price of nontradables by p^N , the agents' budget constraints in all three periods are given by

$$c_{i0}^T + (1 + \tau_i)^{-1}b_{i1} = T_i \quad (2)$$

$$c_{i1}^T(s) + p_1^N(s)c_{i1}^N(s) + b_{i2}(s) = (1 + r)b_{i1} + y_1^T(s) + p_1^N(s)y^N \quad \forall s \quad (3)$$

$$c_{i2}^T(s) + p_2^N(s)c_{i2}^N(s) = (1 + r)b_{i2}(s) + y^T + p_2^N(s)y^N \quad \forall s, \quad (4)$$

where b_{t+1} denotes the bond holdings an agent chooses at the beginning of period t , τ_i denotes the tax rate on date 0 borrowing, and T_i is a lump-sum transfer. Crucially, we impose $\tau_U = 0$, that is, unregulated agents face no tax on borrowing.⁴ To abstract from the distributional side effects of macroprudential policy, we assume that the planner rebates the tax proceeds to the agents who pay the tax.

At date 1, agents are subject to a credit constraint preventing them from borrowing more than a fraction κ of their current income:

$$b_{i2}(s) \geq -\kappa [p_1^N(s)y^N + y_1^T(s)] \quad \forall s. \quad (5)$$

This credit constraint captures in a parsimonious way the empirical fact that income is critical in determining credit market access. The form of this constraint follows [Mendoza \(2002\)](#) and is common in the literature on financial crises and macroprudential policy (e.g., [Bianchi, 2011](#) and [Korinek, 2018](#)).⁵

Agents choose consumption and savings to maximize their utility (1) subject to budget constraints (2), (3), (4), and the credit constraint (5), taking $p_1^N(s)$ and $p_2^N(s)$ as given. Necessary

⁴We assume that unregulated agents are prevented from arbitraging via borrowing abroad and lending domestically to the regulated sphere (e.g., because of technological reasons). Equivalently, we could assume that the tax on borrowing on the regulated sphere applies to both domestic and foreign forms of borrowing.

⁵This collateral constraint can be derived endogenously from limited enforcement assuming that: (i) households can default at the end of the period (before observing next period income realization), (ii) upon default, foreign creditors can seize a fraction κ of the current income and households regain access immediately to credit markets. Non-tradable goods enter the collateral constraint because while foreign creditors do not value non-tradable goods, they can sell it in exchange for tradable goods after seizing these goods after default. The current and not the future price appears in the constraint because the opportunity to default is at the end of the current period, before the realization of future shocks (see [Bianchi and Mendoza, 2018](#)).

and sufficient conditions for optimality are given by

$$p_t^N(s) = \frac{1 - \omega}{\omega} \frac{c_{it}^T(s)}{c_{it}^N(s)}, \quad (6)$$

$$1 = \beta(1 + r)(1 + \tau_i) \mathbb{E}_0 \left[\frac{\omega}{c_{i1}^T(s)} \right], \quad (7)$$

$$\frac{\omega}{c_{i1}^T(s)} \leq \beta(1 + r) \frac{\omega}{c_{i2}^T(s)} \quad \text{with equality if } b_{i2}(s) > -\kappa(p_1^N(s)y^N + y_1^T(s)) \quad (8)$$

for all s . Condition (6) is a static optimality condition equating the marginal rate of substitution between tradable and nontradable goods to their relative price. Conditions (7) and (8) are the Euler equations for bonds at date 0 and date 1. The latter holds with strict inequality if the credit constraint binds, in which case current marginal utility exceeds the expected marginal utility costs from borrowing one unit and repaying next period.

Finally, the government budget constraint says that taxes on borrowing are rebated back to regulated agents:

$$T_R = -\frac{\tau_R}{1 + \tau_R} b_{R1} \quad \text{and} \quad T_U = 0. \quad (9)$$

Definition of competitive equilibrium. Given a tax policy τ_R, T_R , a competitive *regulated* equilibrium of the model is a set of date 0 choices $\{c_{i0}^T, b_{i1}\}_{i \in \{U, R\}}$, date 1 choices $\{c_{i1}^T(s), c_{i2}^T(s), c_{i1}^N(s), c_{i2}^N(s), b_{i2}(s)\}_{i \in \{U, R\}, s \in S}$, and prices $\{p_1^N(s), p_2^N(s)\}_{s \in S}$ such that (1) given prices and taxes, agents' decisions are optimal, (2) the market for the nontradable good clears at all dates, and (3) the government budget constraint holds.

In what follows, we proceed by backward induction. We first analyze the date 1 continuation equilibrium for given date 0 bond choices and then turn to the determination of date 0 borrowing decisions.

2.2 Date 1 continuation equilibrium

To simplify the analysis, we make the following parametric assumptions.

Assumption 1. *The domestic agents' discount factor and the international interest rate satisfy $\beta(1 + r) = 1$.*

This assumption, common in small open economy models, implies that domestic agents are as patient as international investors. A result of this assumption is that there is no intrinsic motivation for consumption tilting in the domestic economy between date 1 and date 2, which simplifies the households' optimization problem.

Assumption 2. *The consumption shares and collateralizable fraction of income are such that $0 < \kappa < \omega/(1 - \omega)$.*

This assumption simplifies the analysis by guaranteeing that there is a unique (continuation) equilibrium. As will become clear below, when Assumption 2 holds, an increase in aggregate consumption by one unit does not relax agents' credit constraint by more than one unit in equilibrium.

For given date 0 savings choices, the agent's date 1 problem conveniently admits a closed-form solution. If an agent i is unconstrained at date 1, his consumption plan for all s is given by

$$c_{i1}^T(s) = c_{i2}^T(s) = \frac{\omega}{1+\beta} we_{i1}(s), \quad \text{and} \quad c_{it}^N(s) = \frac{1-\omega}{1+\beta} \frac{we_{i1}(s)}{p_t^N(s)}, \quad \text{for } t = 1, 2, \quad (10)$$

where $we_{i1}(s)$ is the agent's date 1 lifetime wealth:

$$we_{i1}(s) \equiv (1+r)b_{i1} + y_1^T(s) + p_1^N(s)y^N + \frac{y_2^T + p_2^N(s)y^N}{1+r}.$$

To finance this consumption plan, the agent borrows the shortfall between his expenditures $we_{i1}(s)/(1+\beta)$ and cash on hand $(1+r)b_{i1} + y_1^T(s) + p_1^N(s)y^N$ at date 1:

$$b_{i2}(s) = b_{i2}^{unc}(s) \equiv \frac{\beta}{1+\beta} \left[(1+r)b_{i1} + y_1^T(s) + p_1^N(s)y^N - \frac{y_2^T + p_2^N(s)y^N}{1+r} \right]. \quad (11)$$

The agent is constrained at date 1 if the bond position in (11) violates the credit constraint (5). In this case, he borrows the maximum amount:

$$b_{i2}(s) = b_{i2}^{con}(s) \equiv -\kappa [y_1^T(s) + p_1^N(s)y^N] \quad (12)$$

and chooses a consumption plan given by

$$\begin{aligned} c_{i1}^T(s) &= \omega \tilde{w}e_{i1}(s) & c_{i2}^T(s) &= \omega(1+r) [we_{i1}(s) - \tilde{w}e_{i1}(s)] \\ c_{i1}^N(s) &= (1-\omega) \frac{\tilde{w}e_{i1}(s)}{p_1^N(s)} & c_{i2}^N(s) &= (1-\omega)(1+r) \frac{we_{i1}(s) - \tilde{w}e_{i1}(s)}{p_2^N(s)}, \end{aligned} \quad (13)$$

where $\tilde{w}e_{i1}(s)$ is the agent's date 1 constrained wealth

$$\tilde{w}e_{i1}(s) \equiv (1+r)b_{i1} + (1+\kappa) [y_1^T(s) + p_1^N(s)y^N],$$

which corresponds to the sum of actual date 1 wealth and the maximum amount that can be borrowed.

The nontradable goods market clearing condition is

$$\gamma C_{Ut}^N(s) + (1-\gamma)C_{Rt}^N(s) = y^N, \quad \text{for } t = 1, 2, \quad (14)$$

where uppercase letters with U or R subscripts denote aggregates over an agent's type.

The aggregation of the two sets of agents' intertemporal budget constraints yields the econ-

omy's intertemporal resource constraint:

$$C_1^T(s) + \frac{C_2^T(s)}{1+r} = (1+r)[\gamma B_{U1} + (1-\gamma)B_{R1}] + y_1^T(s) + \frac{y_2^T}{1+r}, \quad (15)$$

where $C_t^T(s) \equiv \gamma C_{Ut}^T(s) + (1-\gamma)C_{Rt}^T(s)$ is aggregate tradable consumption.

Combining the nontradable market clearing condition (14) with the agents' static optimality condition (6) indicates that the equilibrium price of nontradables is proportional to the economy's absorption of tradables:

$$p_t^N(s) = \frac{1-\omega}{\omega} \frac{C_t^T(s)}{y^N}. \quad (16)$$

This condition establishes a positive equilibrium relationship between the price of nontradables and aggregate tradable consumption for a given level of nontradable output. Intuitively, when tradable consumption is high relative to nontradables consumption, the relative price of nontradables has to be high. Given homothetic preferences, a negative shock to the tradable goods endowment y^T generates a decline in the demand for both consumption goods. For a given level of nontradable output - and, by market clearing, nontradable consumption - the equilibrium relative price of nontradables p^N must fall to induce agents to substitute tradable with nontradable consumption. Crucial for our analysis is that a higher aggregate level of debt accumulated at date 0 will similarly imply a lower level of tradable consumption and thus a lower price of nontradables for any income shock at date 1. Through the credit constraint (5), this implies a lower borrowing capacity and a tighter borrowing constraint when this constraint is binding.⁶ Individual agents do not internalize these effects, which generates a scope for welfare-improving macroprudential regulation at date 0, as in Bianchi (2011) and others. In this context, the novelty of our analysis regards the planner's inability to tax or regulate the borrowing decision of a subset of agents.

At date 1, the economy's aggregate state variables are given by the tradable goods endowment $y_1^T(s)$ and by the respective aggregate bond positions of unregulated and regulated agents, B_{U1} and B_{R1} . We denote the functions mapping these state variables into date t aggregate tradable consumption, unregulated agents' tradable consumption, and regulated agents' tradable consumption by $C_t^T(y_1^T(s), B_{U1}, B_{R1})$, $C_{Ut}^T(y_1^T(s), B_{U1}, B_{R1})$, and $C_{Rt}^T(y_1^T(s), B_{U1}, B_{R1})$, respectively. Similarly, we respectively denote the functions mapping the state variables into date t aggregate nontradable consumption, unregulated agents' nontradable consumption, and regulated agents' nontradable consumption by $C_t^N(y_1^T(s), B_{U1}, B_{R1})$, $C_{Ut}^N(y_1^T(s), B_{U1}, B_{R1})$, and $C_{Rt}^N(y_1^T(s), B_{U1}, B_{R1})$. Finally, we denote the pricing function by $p_t^N(y_1^T(s), B_{U1}, B_{R1})$. Depending on which set(s) of agents is (are) credit constrained, the economy can be in four regions at date 1: *cc* where both types of agents are constrained, *cu* where U agents are constrained and R agents

⁶From equation (16), an increase in aggregate consumption raises the price of nontradables by $\frac{\omega}{(1-\omega)y^N}$ and hence raises overall borrowing capacity by $\kappa_{1-\omega} \frac{\omega}{1-\omega}$, which by Assumption 2 is strictly lower than one.

are unconstrained, uc where U agents are unconstrained and R agents are constrained, and uu where both types of agents are unconstrained. In each of these regions, tradable consumption is linear in all three state variables. In particular, date 1 aggregate tradable consumption is increasing in $y_1^T(s)$, B_{U1} , and B_{R1} . As a result, the date 1 price of nontradables is increasing in $y_1^T(s)$, B_{U1} , and B_{R1} . In addition, these two date 1 variables are more sensitive to tradable income when credit constraints are binding. Similarly, C_1^T and p_1^N are more sensitive to B_{U1} and B_{R1} in the regions where the credit constraints are binding. Appendix A formally establishes these results and related ones.

Figure 1 illustrates the sensitivity of the price of nontradables p_1^N with respect to the wealth positions B_{R1} and B_{U1} in the four regions, for a given realization of date 1 tradable income. The downward-sloping light lines represent iso-price curves. Lines farther to the northeast represent higher levels of p_1^N . Naturally, agents are constrained at lower wealth levels. The smaller distance between the iso-price curves in the constrained regions reflects a higher sensitivity of p_1^N in these regions. The intuition is as follows. When the credit constraint does not bind, consumption is increasing in wealth because of a standard permanent income effect. When it does bind, however, the sensitivity is higher because of an additional financial amplification effect working through the price of nontradables. The larger the mass of constrained agents, the stronger this financial amplification effect and thus the stronger the sensitivity of p_1^N to debt.

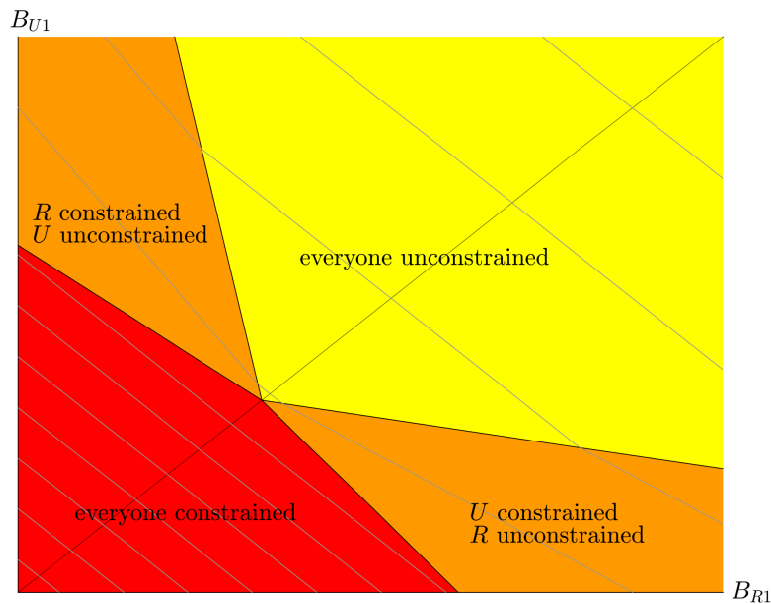


Figure 1: Date 1 price of nontradable good as a function of savings positions B_{R1} and B_{U1} for a given tradable endowment y_1^T .

The properties of the equilibrium nontradable price function discussed above have key implications for the spillover effects of macroprudential policy onto the unregulated sphere of the economy. In particular, the increasingness of p_1^N in B_{R1} means that the date 1 total income at

market prices and the date 1 borrowing capacity of unregulated agents are both increasing in the date 0 savings of the regulated agents. This in turn implies that for a given level of debt of unregulated agents, the consumption of these agents is higher the lower the debt of regulated agents. This is illustrated in Figure 2, which represents unregulated agents' date 1 tradable consumption C_{U1}^T as a function of the realization of tradable income y_1^T , for two different regulated savings levels $\bar{B}_{R1} < \tilde{B}_{R1}$ and a given unregulated savings level $\bar{B}_{U1} = \tilde{B}_{U1}$. The light curve represents the consumption function for a high level of regulated agents' debt (i.e., high B_{R1}), while the dark curve represents the function for a lower level of regulated agents' debt (i.e., a lower B_{R1}). The figure shows that a lower level of regulated agents' debt has several general equilibrium effects on unregulated agents' consumption. First, by increasing total income evaluated at market prices, it shifts consumption up when all agents are unconstrained. Second, by propping up collateral value, it shifts the region where unregulated agents become constrained to the left. Third, and maybe more subtly, it reduces the sensitivity of (unregulated agents') consumption to income for intermediate levels of income at which unregulated agents are constrained but regulated agents are not.

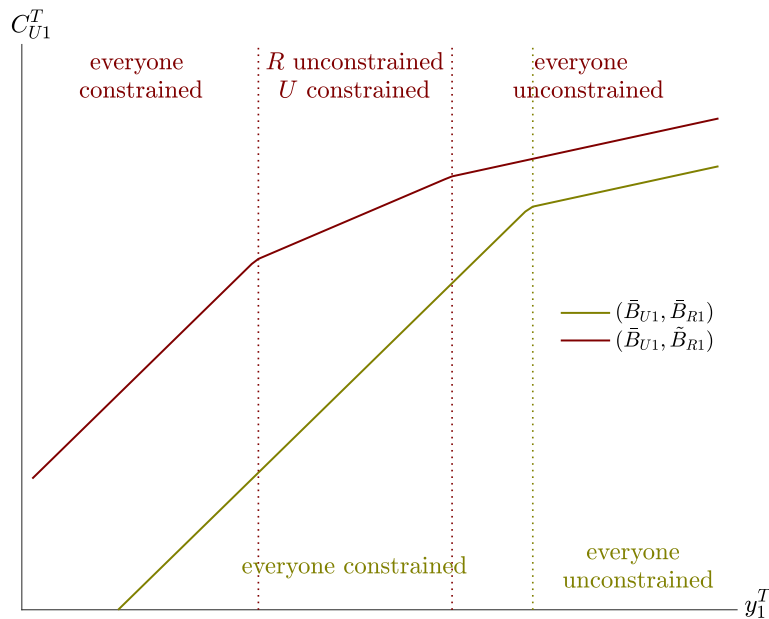


Figure 2: Unregulated agents' date 1 consumption as function of the tradable endowment y_1^T , for given savings pairs $(\bar{B}_{U1}, \bar{B}_{R1})$ and $(\tilde{B}_{U1}, \tilde{B}_{R1})$, with $\bar{B}_{U1} = \tilde{B}_{U1} < \tilde{B}_{R1}$.

The general equilibrium effects just described of regulated agents' borrowing on unregulated agents' date 1 consumption profile naturally translate into spillovers from regulated agents' borrowing into unregulated agents' borrowing at date 0. We elaborate on these in Section 2.4.

At this stage, we can define the date 1 value in equilibrium of an agent of type i as a function

of the aggregate state variables of the economy as

$$V_{i1}(y_1^T(s), B_{U1}, B_{R1}) = \sum_{t=1,2} \beta^{t-1} \left[\ln (C_{it}^T(y_1^T(s), B_{U1}, B_{R1}))^\omega + \ln (C_{it}^N(y_1^T(s), B_{U1}, B_{R1}))^{1-\omega} \right]. \quad (17)$$

This value function will be useful for the normative analysis coming up in Section 2.5.

2.3 Date 0 unregulated equilibrium

We refer to the competitive equilibrium that prevails when the tax τ_R is set to zero as the *unregulated* equilibrium. The unregulated equilibrium naturally displays symmetric borrowing choices ($B_{U1} = B_{R1} \equiv B_1^{ue}$) and is characterized by the following Euler equation:

$$1 = \mathbb{E}_0 \left[\frac{\omega}{C_1^T(y_1^T(s), B_1^{ue}, B_1^{ue})} \right]. \quad (18)$$

Lemma 1. *The date 0 competitive equilibrium exists and is unique.*

2.4 Equilibrium with exogenous tax

To lay the groundwork for our analysis of optimal macroprudential policy in the presence of leakages, we start by characterizing the private sector's response to an exogenous tax on borrowing. In a second step, we will then solve for the optimal tax chosen by a benevolent government.

A competitive regulated equilibrium can be conveniently characterized by a sole pair of Euler equations,

$$1 = \mathbb{E}_0 \left[\frac{\omega}{C_{U1}^T(y_1^T(s), B_{U1}, B_{R1})} \right], \quad (19)$$

$$\frac{1}{1 + \tau_R} = \mathbb{E}_0 \left[\frac{\omega}{C_{R1}^T(y_1^T(s), B_{U1}, B_{R1})} \right], \quad (20)$$

where (19) and (20) are the respective Euler equations for date 0 borrowing of unregulated and regulated agents.

Given a value for B_{R1} , (19) implicitly defines the equilibrium borrowing choice of unregulated agents. Likewise, for a given value of B_{U1} and a given tax τ_R , (20) implicitly defines the equilibrium borrowing choice of regulated agents. We can thus formally define the following equilibrium responses of the two sectors.

Definition 1 (Equilibrium borrowing responses). $B_{U1} = \phi_U(B_{R1})$ denotes the equilibrium borrowing responses of U agents to the borrowing of R agents, as implicitly defined by (19). Similarly, $B_{R1} = \phi_R(B_{U1}, \tau_R)$ denotes the equilibrium borrowing responses of R agents to the borrowing of U agents and a tax rate, as implicitly defined by (20).

To understand the effect of macroprudential policy with leakages, a key issue is the impact

of a change in borrowing by one type of agent on the other agents' borrowing choice. Our next result characterizes such responses.

Proposition 1 (Substitutability in borrowing decisions). *For a given tax rate, the equilibrium borrowing of unregulated agents is decreasing in the amount of borrowing of R agents and vice versa; that is, $\phi'_U(B_{R1}) < 0$ and $\partial\phi_R(B_{U1}, \tau_R)/\partial B_{U1} < 0$.*

Proposition 1 establishes that borrowing decisions by the two sets of agents are akin to strategic substitutes. The less regulated agents borrow, the more unregulated agents find it optimal to borrow (and vice versa). This result follows naturally from our discussion of how unregulated agents' date 1 consumption depended on regulated agents' borrowing in the context of Figure 2.⁷ Lower borrowing by regulated agents at date 0 shifts unregulated agents' consumption up for any realization of the date 1 endowment, notably because it supports higher nontradable goods prices and thus uniformly relaxes everyone's date 1 borrowing constraint. Higher date 1 (tradable) consumption, and thus lower date 1 marginal utility, for a given level of B_{U1} , in turn induces unregulated agents to increase borrowing. The exact same logic applies to regulated agents' response to unregulated agents' borrowing.

While the substitutability emphasized in Proposition 1 is key to grasping our leakage phenomenon, our ultimate interest lies in understanding how both sectors react to macroprudential policy. Our next proposition describes how date 0 borrowing by the two sets of agents and date 1 borrowing capacity respond to changes in the tax rate.

Proposition 2 (Positive effect of small tax). *Starting from the unregulated equilibrium, imposing a small tax leads to strictly less borrowing by regulated agents, strictly more borrowing by unregulated agents, and to an unambiguously larger borrowing capacity at date 1.*

Quite intuitively, Proposition 2 states that a tax on debt generates a decrease in regulated agents' date 0 borrowing. More importantly, and consistent with Proposition 2, it also says that a larger tax causes an increase in unregulated agents' date 0 borrowing.

Figure 3 illustrates these results by representing the equilibrium response functions of the two sets of agents in the (B_{R1}, B_{U1}) space. The solid line is the equilibrium response of unregulated agents, while the other two lines are the equilibrium responses of regulated agents associated with a zero tax (dash-dotted line) and a positive tax (dashed line). The intersection between the unregulated agents' equilibrium response and the regulated agents' equilibrium response associated with a zero tax coincides by definition with the unregulated equilibrium. A positive tax causes a shift of the regulated agents' equilibrium response to the right: for a given B_{U1} choice, regulated agents respond to the tax by borrowing less (it makes borrowing more costly). But unregulated agents respond to this lower borrowing by R agents by borrowing more themselves. This extra borrowing by U agents in turn induces R agents to borrow even less. And U agents

⁷Note that this discussion focused on the effect of regulated agents' borrowing on unregulated agents' consumption. But a similar logic applies for the effect of unregulated agents' borrowing on regulated agents' consumption.

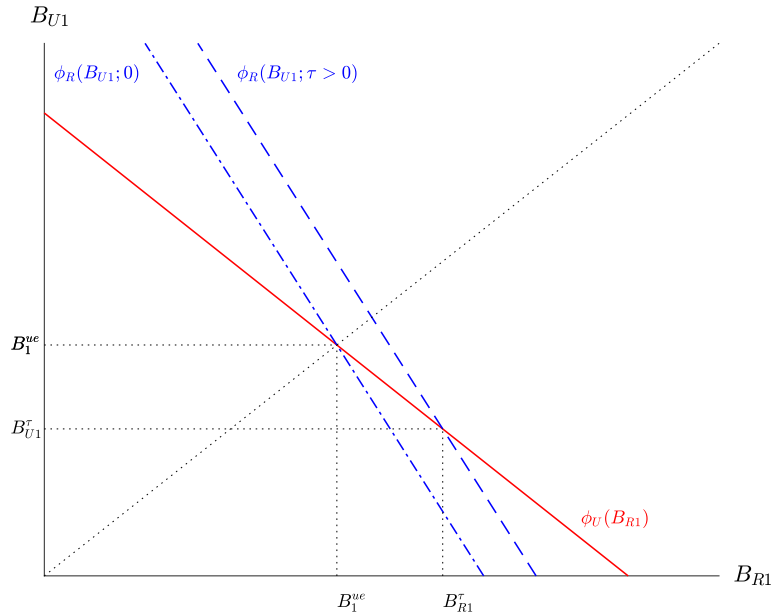


Figure 3: Equilibrium response functions of regulated and unregulated agents in equilibrium with exogenous tax ($0 < \gamma < 1$).

again respond by borrowing even more. This “process” continues until equilibrium is reached at point $(B_{R1}^{\tau}, B_{U1}^{\tau})$. The equilibrium response of unregulated agents can thus be thought of as the set of (B_{R1}, B_{U1}) combinations that a government could achieve by varying the tax τ_R .

Finally, Proposition 2 indicates that the effect of a larger tax on the *date 1* borrowing capacity is positive. This effect can be traced back to the net effect of an increase in regulated agents’ wealth on the date 1 price of nontradable goods:

$$\frac{dp_1^N}{dB_{R1}} = \frac{\partial p_1^N}{\partial B_{R1}} + \underbrace{\frac{\partial p_1^N}{\partial B_{U1}} \frac{\partial B_{U1}}{\partial B_{R1}}}_{\text{Leakage effect}}. \quad (21)$$

That is, the direct effect of the tax on the date 1 price of nontradables via regulated agents’ borrowing (first term in (21)) necessarily dominates its indirect effect via unregulated agents’ borrowing (second term in (21)), resulting in a positive net effect. This suggests that, at least locally, leakages may reduce the effectiveness of macroprudential policy by making future borrowing capacity less responsive to a tax on current borrowing, but are not powerful enough to overturn its effect.

2.5 Optimal macroprudential policy

The preceding section characterized the private sector’s response to an exogenous tax. Building on this positive analysis, we now take a normative perspective and endogenize the level of the tax as the outcome of an optimal policy problem.

Before turning to a formal analysis of optimal policy, we find it useful to highlight the scope for welfare improvements of macroprudential policy with the following preliminary result.

Proposition 3 (Welfare effect of small tax). *If borrowing constraints bind with positive probability in the unregulated equilibrium, a small positive tax is welfare improving for all agents.*

This result, based on a perturbation argument, indicates that in spite of leakages, it is always desirable for the planner to impose a small tax on regulated agents. According to Proposition 2, such a tax leads to less borrowing by regulated agents, more borrowing by unregulated agents, and a higher date 1 borrowing capacity. The only first-order effect on welfare (for both agents) arises from the relaxation of the borrowing constraint at date 1 in states of nature where this constraint binds and is positive. All other effects are of second order. This suggests that there exists potential welfare gains from macroprudential policy despite leakages.

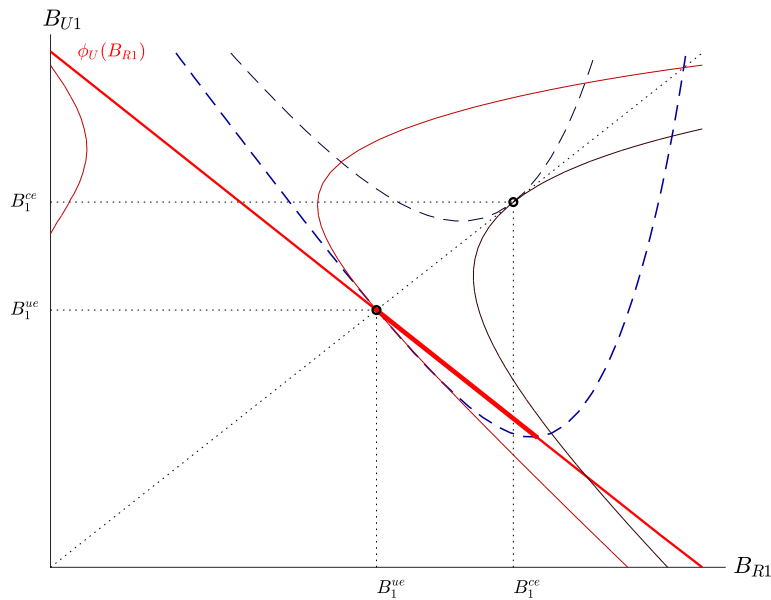


Figure 4: Contract curve and implementability constraint

Figure 4 plots the unregulated agents' equilibrium response function $\phi_U(B_{R1})$ together with both types of agents' iso-utility curves passing through the unregulated equilibrium (B_1^{ue}, B_1^{ue}) . Proposition 3 implies that the segment of the unregulated agents' equilibrium response function situated immediately to the right of the unregulated equilibrium necessarily lies in the lens formed by the two types of agents' iso-utility curves passing through the unregulated equilibrium.

Turning to optimal policy, we consider a planner maximizing a weighted sum of the agents' utility subject to the date 1 and date 2 objects being the outcomes of a continuation equilibrium, and subject to the unregulated agents' private optimality condition for borrowing (19). The

planner's problem in primal form is thus given by

$$\max_{B_{R1}, B_{U1}} \gamma \delta [-B_{U1} + \beta \mathbb{E}_0 V_{U1}(y_1^T(s), B_{U1}, B_{R1})] + (1 - \gamma) [-B_{R1} + \beta \mathbb{E}_0 V_{R1}(y_1^T(s), B_{U1}, B_{R1})] \quad (22)$$

subject to

$$1 = \mathbb{E}_0 \left[\frac{\omega}{C_{U1}^T(y_1^T(s), B_{U1}, B_{R1})} \right],$$

where the functions V_{U1} and V_{R1} are defined in (17) and δ is a relative Pareto weight on unregulated agents. When $\delta = 1$, the planner's objective becomes utilitarian, with the weight on agents' welfare being equal to their share in the population.

Before formally characterizing the optimal policy, let us stress how the planner's problem differs from typical planning problems in environments with pecuniary externalities. As is standard, the planner effectively controls B_{R1} by means of the tax on borrowing. However, here the choice of B_{U1} is up to unregulated agents. As a result, the planner is constrained to choosing a pair (B_{R1}, B_{U1}) from within the equilibrium best response function of unregulated agents, as depicted by the solid line in Figure 4.⁸ Absent this constraint, the solution to the planning problem would be the constrained-efficient allocation, depicted as the point (B_1^{ce}, B_1^{ce}) in Figure 4.

The planner's optimal choice of B_{R1} is characterized by the following generalized Euler equation (GEE):

$$1 = \mathbb{E}_0 \frac{\omega}{C_{R1}^T} + \kappa \mathbb{E}_0 \left[\left(\mu_{R1} + \frac{\delta \gamma}{1 - \gamma} \mu_{U1} \right) \frac{dp_1^N}{dB_{R1}} \right] + \gamma \beta \mathbb{E}_0 \sum_{t=1}^2 \left[\left(\delta \frac{\omega}{C_{Ut}^T} - \frac{\omega}{C_{Rt}^T} \right) (C_{Rt}^N - C_{Ut}^N) \frac{dp_1^N}{dB_{R1}} \right] \quad (23)$$

where $\mu_{i1} \equiv \omega/C_{i1}^T - \omega/C_{i2}^T \geq 0$ are the shadow costs associated with the credit constraints at date 1, and dp_1^N/dB_{R1} is given by (21).

This GEE equates the marginal costs from reducing borrowing and consumption today with the marginal benefits of having lower debt tomorrow. It resembles the private Euler equation (7), but contains additional terms reflecting the planner's internalization of pecuniary externalities and spillovers to unregulated agents. The left-hand side is the same and corresponds to the marginal utility cost of reducing consumption by one unit. Given linear utility, this term is a constant equal to one. The first term on the right-hand side of (23) corresponds to the private marginal utility cost from borrowing one less unit and raising consumption tomorrow, also present in (7).

The second term in (21) constitutes the pecuniary externality: the planner internalizes that a lower level of debt leads to an increase in the price of nontradable goods at date 1 and a

⁸Naturally, the unregulated equilibrium belongs to the set of feasible (B_{R1}, B_{U1}) pairs.

relaxation of collateral constraints for both regulated and unregulated agents at that date. The benefit of relaxing the collateral constraint is an average of the Lagrange multipliers, weighted by the relevant Pareto weights. A similar expression is common in the normative analysis of models with credit constraints linked to market prices (e.g., Bianchi, 2011). However, unlike in models with perfect financial regulation enforcement, it here embeds the unregulated agents' response to the borrowing choice of the planner for the regulated agents, through the negative partial derivative $\partial B_{U1}/\partial B_{R1}$ present in the price derivative term dp_1^N/dB_{R1} . This leakage effect lowers the marginal value of saving on behalf of regulated agents for the planner and therefore reduces the wedge between the planner's and private agent's perceived benefit of having lower debt. The final term of the GEE reflects the planner's marginal value of the wealth redistribution induced by a higher price of nontradables.⁹

The discrepancy between the planner's GEE (23) and the regulated agent's private Euler equation for debt (7) calls for a Pigouvian tax on debt, as in the existing literature. The next proposition establishes that as long as the planner can tax a subset of agents, irrespective of how large (or small) this subset is, he will choose to set a strictly positive tax whenever the credit constraint binds with strictly positive probability.

Proposition 4 (Optimal tax). *The optimal tax is given by*

$$\tau_R = \frac{\beta \mathbb{E}_0 \left[\left(\mu_{R1} + \frac{\delta \gamma}{1-\gamma} \mu_{U1} \right) \kappa y^N \frac{dp_1^N}{dB_{R1}} \right] + \gamma \beta \mathbb{E}_0 \left[\sum_{t=1}^2 \left(\delta \frac{\omega}{C_{Ut}^T} - \frac{\omega}{C_{Rt}^T} \right) (C_{Rt}^N - C_{Ut}^N) \frac{dp_t^N}{dB_{R1}} \right]}{\mathbb{E}_0 \left[\frac{\omega}{C_{R1}^T} \right]}. \quad (24)$$

As a result, the optimal tax on borrowing is strictly positive whenever the credit constraint binds with positive probability in the unregulated equilibrium.

The optimal tax captures the uninternalized marginal costs of borrowing from the GEE (23). The optimal tax expression suggests an ambiguous effect of leakages on the magnitude of the tax. Two key forces work in opposite directions. First, the leakage term in (21) reduces the magnitude of the price derivative term dp_1^N/dB_{R1} , calling for a lower tax. Second, leakages lead to higher shadow values of relaxing the collateral constraint for unregulated agents μ_{U1} , calling for a higher tax. In other words, the macroprudential policy's reduced effectiveness speaks in favor of a lower tax, while larger benefits from financial stabilization speak in favor of a higher tax, resulting in an ambiguous total effect. Similarly, the relationship between the size of the unregulated sector γ and the optimal tax is a priori ambiguous.

⁹A higher price of nontradables redistributes wealth from the net buyer of nontradables to the net sellers of nontradables.

2.6 Endogenous fraction of unregulated agents

So far, our analysis of macroprudential policy with leakages has taken the size of the unregulated sector as exogenously given. In this section, we present a simple way to endogenize this variable with a model of free entry into the unregulated sphere.

Our baseline model is extended by adding a period prior to date 0, labeled date -1 , when households choose whether they want to join the unregulated sector. To do so, they must pay a linear tax circumvention cost φ randomly distributed over the population and realized before the decision to circumvent is made.

To choose whether to join the unregulated sector, agents conjecture a tax τ_R^c to be set by the government at date 0. It follows that an agent joins the unregulated sphere when

$$W_U(\tau^c) - \varphi^i > W_R(\tau^c)$$

and remains in the regulated sphere otherwise.

Assuming that the planner is unable to commit over the tax at date -1 , the size of the unregulated sector is a fixed point.¹⁰ For a given conjecture of the tax, households choose whether they want to be unregulated. In turn, given the implied size of the unregulated sector, the government chooses the tax expected by agents at date -1 . It is easy to see that any γ can be rationalized by some distribution of idiosyncratic costs. Following that logic, our insights remain unchanged once we endogenize γ .

2.7 Insights from three-period model

This section presented a highly stylized model of imperfectly enforced macroprudential policy, where the inherent motivation for a tax on borrowing derived from a pecuniary externality caused by financial constraints linked to a market price. The key prediction of the model is that in response to a tax on borrowing for the regulated sphere, borrowing by the unregulated sphere increases. The main normative insight is that this leakage phenomenon exerts two counteracting forces on the magnitude of the optimal macroprudential tax on borrowing. On the one hand, the leakage makes the macroprudential tax on the regulated sphere less effective because the reduction in the regulated sphere's indebtedness is partially offset by an increase in borrowing by

¹⁰Under commitment, there would be an additional effect. Defining $\varphi^*(\tau_R^c)$ as the threshold satisfying $W^U(\tau_R^c) - \varphi^*(\tau_R^c) = W^R(\tau_R^c)$ and denoting the cumulative density of φ by F , the planner would solve

$$\begin{aligned} \max_{B_{R1}, B_{U1}} & - \int_0^{\varphi^*(\tau)} \varphi dF + (1 - F(\varphi^*(\tau))) \delta [-B_{U1} + \beta \mathbb{E}_0 V_{U1}(y_1^T(s), B_{U1}, B_{R1})] \\ & + F(\varphi^*(\tau)) [-B_{R1} + \beta \mathbb{E}_0 V_{R1}(y_1^T(s), B_{U1}, B_{R1})]. \end{aligned} \quad (25)$$

The expression for the optimal tax would internalize the effect of the tax on the extent of circumvention, as captured by the term $F(\varphi^*(\tau))$.

the unregulated sphere. On the other hand, the leakages make the macroprudential tax introduce a new distortion that takes the form of an even more excessive indebtedness of the unregulated sphere. Correcting this distortion requires reducing the economy's indebtedness further and therefore calls, paradoxically, for even tighter borrowing restrictions on the regulated sphere.

3 Quantitative Model

In this section, we embed the leakage phenomenon into a canonical quantitative model of financial crisis. The goal is to assess the extent to which leakages limit the ability of macroprudential regulation to reduce the exposure to financial crisis and to study how leakages affect the optimal policy design. From a theoretical perspective, an additional element emerges in the infinite horizon model. Given the forward-looking nature of the unregulated agents' problem, these agents' borrowing decision depends not only on current regulation but also on their expectation of future regulation. As a result, a new time inconsistency problem emerges that would not be present with perfect enforcement of regulation.

3.1 Preferences and constraints in the infinite horizon model

As in the three-period model, there are two types of agents with identical preferences and endowments, who only differ on whether they are subject to borrowing taxes. Preferences are given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}), \quad (26)$$

where $u(\cdot)$ is a standard concave, twice continuously differentiable function that satisfies the Inada condition. The consumption basket c is an Armington-type CES aggregator with elasticity of substitution $1/(\eta + 1)$ between tradable goods c^T and nontradable goods c^N , given by

$$c = \left[\omega (c^T)^{-\eta} + (1 - \omega) (c^N)^{-\eta} \right]^{-\frac{1}{\eta}}, \eta > -1, \omega \in (0, 1).$$

In each period t , agents receive endowments of tradable goods y_t^T and nontradable goods y_t^N and choose a one-period non-state-contingent bond denominated in units of tradables. The vector of endowments $\mathbf{y} \equiv (y^T, y^N)' \in Y \subset R_{++}^2$ follows a first-order Markov process. The agents' budget constraints and credit constraints are given by

$$\frac{b_{it+1}}{R(1 + \tau_{it})} + c_{it}^T + p_t^N c_{it}^N = b_{it} + y_t^T + p_t^N y_t^N + T_{it} \quad (27)$$

and

$$b_{it+1} \geq -\kappa (p_t^N y_t^N + y_t^T). \quad (28)$$

As in the three-period model, we assume that $\tau_{Ut} = T_{Ut} = 0$ for all $t \geq 0$. Given a tax

policy $\{\tau_{Rt}, T_{Rt}\}_{t \geq 0}$ and initial levels of debt b_{U0}, b_{R0} , a competitive equilibrium is defined as a stochastic sequence of prices $\{p_t^N\}_{t \geq 0}$ and households' policies $\{c_{it}^T, c_{it}^N, b_{it+1}\}_{t \geq 0, i \in \{U, R\}}$ such that (i) households maximize (26) subject to sequences of budget constraints (27) and credit constraints (28), (ii) the market clears for nontradable goods ($\gamma c_{Ut}^N + (1 - \gamma)c_{Rt}^N = y_t^N$), and (iii) the government budget constraint holds $T_R = -\frac{b_{Rt}}{R} \left(\frac{\tau}{1+\tau}\right)$.¹¹

3.2 Optimal time-consistent regulated equilibrium

As in Section 2.5, we consider the problem of a planner choosing the tax policy that delivers the highest welfare in the regulated equilibrium.

We assume that the planner makes decisions sequentially and without commitment, and we study Markov-perfect equilibria. Focusing on a discretionary regime is useful for our purpose given our goal of studying how leakages can undermine the effectiveness of regulation. We let $X = \{B_U, B_R, y^T, y^N\}$ denote the aggregate state vector of the economy, $\mathcal{B}_R(X)$ denote the policy rule for regulated bond holdings of future planners that the current planner takes as given, and $\{\mathcal{B}_U(X), \mathcal{C}_R^T(X), \mathcal{C}_U^T(X), \mathcal{C}_R^N(X), \mathcal{C}_U^N(X), \mathcal{P}^N(X)\}$ denote the associated recursive functions returning unregulated agents' bond holdings, consumption allocations, and the price of nontradables under this policy rule.¹²

The forward-looking nature of unregulated agents' borrowing decisions introduces a time consistency problem. To see why, consider the Euler equation of unregulated agents when it holds with equality:

$$u_T(c_U^T, c_U^N) = \beta R E u_T(c_U^T(X'), c_U^N(X')). \quad (29)$$

As was the case of constraint (19) in the three-period model, this constraint is a key implementability constraint for the government in the infinite horizon model. It captures the spillover effects from the planner's debt choice for R agents onto U agents' debt choice. However, in the infinite horizon context where the planner regulates the economy in every period, this implementability constraint depends on next period's regulatory policy. For example, a policy that induces low consumption of U agents tomorrow (through a low tax on R agents in $t + 1$) will indirectly push down these agents' consumption today and moderate the overborrowing externality in the present. Moreover, if unregulated agents expect loose regulation in the future, they have incentives to accumulate more precautionary savings today and borrow less. Tomorrow, however, the planner acting without commitment will not internalize the benefits of such a loose tax policy over previous periods.¹³

¹¹The definition of equilibrium in recursive form is given in Appendix C.

¹²Notice that by a form of block recursivity, once the policy for regulated bond holdings is chosen, the rest of the equilibrium objects can be obtained from the implementability constraints.

¹³The time-inconsistency problem of macroprudential policy that we highlight is distinct from the one in Bianchi and Mendoza (2018). In that paper, a time-inconsistency problem arises because of the presence of asset prices, a forward-looking object, in collateral constraints. Here instead, it is due to imperfect regulation enforcement.

Following again the primal approach adopted in the three-period model context (see Section 2.5), assuming a utilitarian objective, the optimal time-consistent (TC) planner's problem can be described by the following Bellman equation:

$$\mathcal{V}(X) = \max_{\{c_i^T, c_i^N, b_i\}_{i \in \{U, R\}} \cdot p^N} \gamma u(c(c_U^T, c_U^N)) + (1 - \gamma)u(c(c_R^T, c_R^N)) + \beta \mathbb{E}\mathcal{V}(X') \quad (\text{Optimal TC})$$

subject to

$$c_i^T + p^N c_i^N + \frac{b_i'}{R} = b_i + y^T + p^N y^N \quad \text{for } i \in \{U, R\}$$

$$b_i' \geq -\kappa (p^N y^N + y^T) \quad \text{for } i \in \{U, R\}$$

$$c_i^N = \frac{c_i^T}{\gamma c_U^T + (1 - \gamma)c_R^T} y^N \quad \text{for } i \in \{U, R\}$$

$$p^N = \frac{1 - \omega}{\omega} \left(\frac{\gamma c_U^T + (1 - \gamma)c_R^T}{y^N} \right)^{\eta+1}$$

$$u_T(c_U^T, c_U^N) \geq \beta R \mathbb{E} u_T(\mathcal{C}_U^T(X'), \mathcal{C}_U^N(X'))$$

$$[b_U' + \kappa (p^N y^N + y^T)] \times [u_T(c_U^T, c_U^N) - \beta R \mathbb{E} u_T(\mathcal{C}_U^T(X'), \mathcal{C}_U^N(X'))] = 0.$$

A **Markov equilibrium** is defined by policy functions $\{\mathcal{B}_R(X)\mathcal{B}_U(X), \mathcal{C}_R^T(X), \mathcal{C}_U^T(X), \mathcal{C}_R^N(X), \mathcal{C}_U^N(X)\}$, a value function $\mathcal{V}(X)$, and a pricing function $\mathcal{P}^N(X)$ such that the value function and policy functions solve (Optimal TC) given perceived policies $\{\mathcal{B}_i(X), \mathcal{C}_i^T(X), \mathcal{C}_i^N(X)\}$ for $i \in \{U, R\}$ and $\mathcal{P}^N(X)$.

3.3 Calibration

The calibration follows Bianchi (2011). The time period is one year. A first subset of parameters is set independently using standard values from the literature: $\sigma = 2$, $r = 0.04$, $1/(\eta + 1) = 0.83$, and the endowment process is estimated based on the HP-filtered component of tradable and nontradable GDP for Argentina. Assuming a first-order bivariate autoregressive process: $\ln \mathbf{y} = \rho \ln \mathbf{y}_{t-1} + \varepsilon_t$, where $\varepsilon_t = (\varepsilon_t^T, \varepsilon_t^N)' \sim N(0, \Sigma_\varepsilon)$, we obtain the estimates

$$\rho = \begin{bmatrix} 0.901 & -0.453 \\ 0.495 & 0.225 \end{bmatrix} \quad \Sigma_\varepsilon = \begin{bmatrix} 0.00219 & 0.00162 \\ 0.00162 & 0.00167 \end{bmatrix}.$$

The second subset of parameters $\{\beta, \omega, \kappa\}$ is set to match Argentina's average net foreign asset position, the share of nontradable output in Argentina, and the average frequency of financial crises for emerging markets.¹⁴ This yields $\beta = 0.91$, $\omega = 0.31$, $\kappa = 0.32$.

Other contributions studying optimal policy problems with Markov perfect equilibria include Klein, Quadrini and Rios-Rull (2005), Klein, Krusell and Rios-Rull (2008), and Debortoli, Nunes and Yared (2017).

¹⁴In both the model and the data, financial crises are defined as episodes in which the current account increases by more than one standard deviation.

Finally, we solve our regulated equilibrium for different values of γ . For the most part, we focus on values of γ ranging from 0 to 0.5. A value of 0.5 entails a substantial amount of leakages by which 50% of the economy can evade regulation. Moreover, a value of $\gamma = 0.5$ is the value at which the losses from a dispersion in consumption across agents are given the highest weight by the planner.

Table 1: Calibration

	Value	Source/Target
Interest rate	$r = 0.04$	Standard value
Risk aversion	$\sigma = 2$	Standard value
Elasticity of substitution	$1/(1 + \eta) = 0.83$	Conservative value
Weight on tradables in CES	$\omega = 0.31$	Share of tradable output=32%
Discount factor	$\beta = 0.91$	Average NFA-GDP ratio = -29%
Credit coefficient	$\kappa^T = 0.32$	Frequency of crises = 5.5%
Size of unregulated sector	$\gamma = [0, 0.5]$	Baseline range

3.4 Numerical solution

The computation of the optimal regulated equilibrium follows a nested fixed point algorithm, common to those used in studies of Markov perfect equilibria (e.g. [Bianchi and Mendoza, 2018](#)). For a given conjectured policy followed by governments in the future, we solve for the current optimal policy using value function iteration. Using this solution, we update our conjectured policy. We iterate until the optimal policy coincides with the conjectured policy to obtain a Markov-perfect equilibrium. Details are provided in [Appendix D](#).

3.5 Overborrowing and leakages

We start our quantitative analysis by looking at how the distribution of debt of regulated and unregulated agents differs across regimes. To show this, we conduct a 10,000-period simulation for the unregulated equilibrium, the constrained-efficient allocation and the regulated equilibrium for $\gamma = 0.5$ and provide a scatterplot of the bond positions.

Panels (a) and (b) of [Figure 5](#) correspond to the scatterplots of bond positions in the unregulated equilibrium and constrained-efficient allocation. Vertical and horizontal lines show sample averages. Because there is no distinction between U and R agents in these economies, all the points line up on the 45-degree line. One can see, as expected, that the unregulated equilibrium displays simulations with higher levels of debt, which in turn are associated with a larger frequency and severity of financial crises.

Panel (c) of [Figure 5](#) displays the case of the regulated equilibrium for $\gamma = 0.5$. In this case, most pairs of bond positions are located below the 45-degree line, indicating higher levels of debt

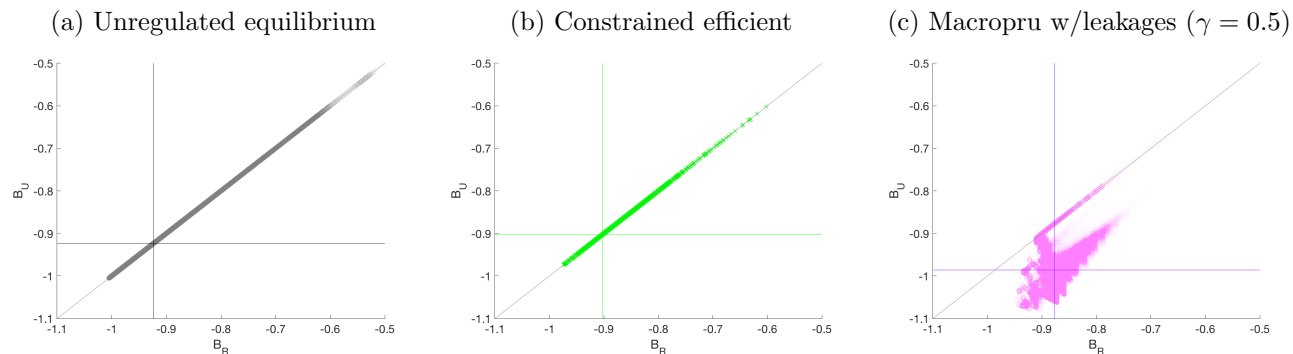


Figure 5: Scatterplots of debt positions in unregulated equilibrium, constrained-efficient allocation, and macroprudential policy with leakages.

for unregulated agents than for regulated agents. Interestingly, many simulation periods display levels of debt for unregulated agents that are much higher than the maximum values of debt observed in the unregulated equilibrium. This is the leakage effect at play: regulation worsens the overborrowing problem for unregulated agents. Conversely, regulated agents' borrowing in the regulated equilibrium with leakages is lower on average than in the constrained-efficient equilibrium. This suggests that the planner (at least partially) compensates for the unregulated agents' extra borrowing by commanding less borrowing for regulated agents.

3.6 Frequency and severity of crises

Next, we study the extent to which leakages undermine the effectiveness of regulation at reducing the vulnerability to financial crises. We define financial crises as episodes in which the current account increases by more than one standard deviation (or, equivalently, credit falls by more than one standard deviation). Based on this definition, we study how the probability of crises varies in regulated equilibria associated with a range of values for γ , and compare the severity of crises in a regulated equilibrium where $\gamma = 0.5$ to the severity in the constrained-efficient allocation and unregulated equilibrium.

Figure 6 shows how the frequency of financial crises changes with the size of the unregulated sphere γ . In the absence of leakages (i.e., when $\gamma = 0$), the frequency of crises is about 0.5%, which is about 1/10th of the frequency of crises in the unregulated equilibrium. As expected, the frequency of crises increases with γ . Quite strikingly, however, the increase is very modest even for values of γ as large as 0.5. This suggests that as γ increases, the planner adjusts its desired borrowing for regulated agents to offset the effect of leakages and achieve a given level of financial stability.

To study how leakages alter the government's ability to reduce the severity of financial crises using macroprudential regulation, we construct a comparable event analysis following a procedure similar to [Bianchi and Mendoza \(2018\)](#). First, we simulate the decentralized equilibrium for a

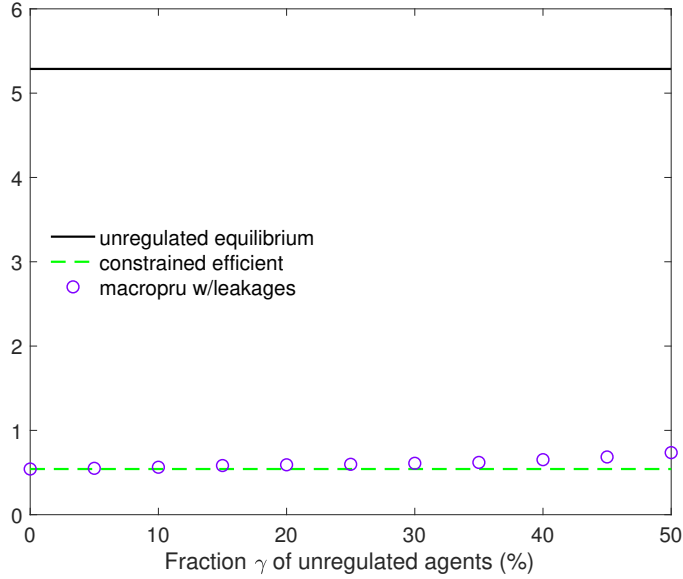


Figure 6: Long-run frequency of financial crises as a function of γ .

large number of periods, identify all the financial crisis episodes, and construct 11-year window events centered on the financial crisis episodes. Second, we take the average of key variables across the window period for the decentralized equilibrium. Third, we feed in the initial state and shock sequence that characterizes all financial crises in the unregulated equilibrium to the policy functions of the regulated equilibrium. We do this for two degrees of leakages: $\gamma = 0$, which corresponds to the constrained-efficient allocation, and $\gamma = 0.5$. Finally, we average the key variables across the window period for the regulated equilibria. This experiment allows us to do a counterfactual analysis that highlights how leakages lead to different financial crises dynamics, controlling for the same sequence of shocks and the same initial states.

Figure 7 shows the results of these simulations. In the top panels, we plot the income shocks, the current account to GDP ratio, and the real exchange rate. In the bottom panels, we show the debt of regulated agents, the debt of unregulated agents, and the optimal tax. All the plotted paths correspond to averages across all the simulation samples from the event analysis. The unregulated equilibrium (solid line) clearly displays a larger decline in credit and a larger current account reversal, as well as a larger collapse in the real exchange rate (defined as the inverse of the price of the composite good). The crises are preceded by increases in the amount of credit and negative income shocks, and are triggered on impact by income shocks that are about 1.5 standard deviations on average. In contrast, the constrained-efficient equilibrium (dashed line) displays a much smaller decline in credit (1% percent versus 10% for the unregulated equilibrium) and a much smaller decline in the real exchange rate. Note that these differences in the event dynamics emerge despite the two economies having the same initial conditions and being subject to the same shock sequence.

In terms of aggregate variables, one can see that the regulated equilibrium for $\gamma = 0.5$ (macro-

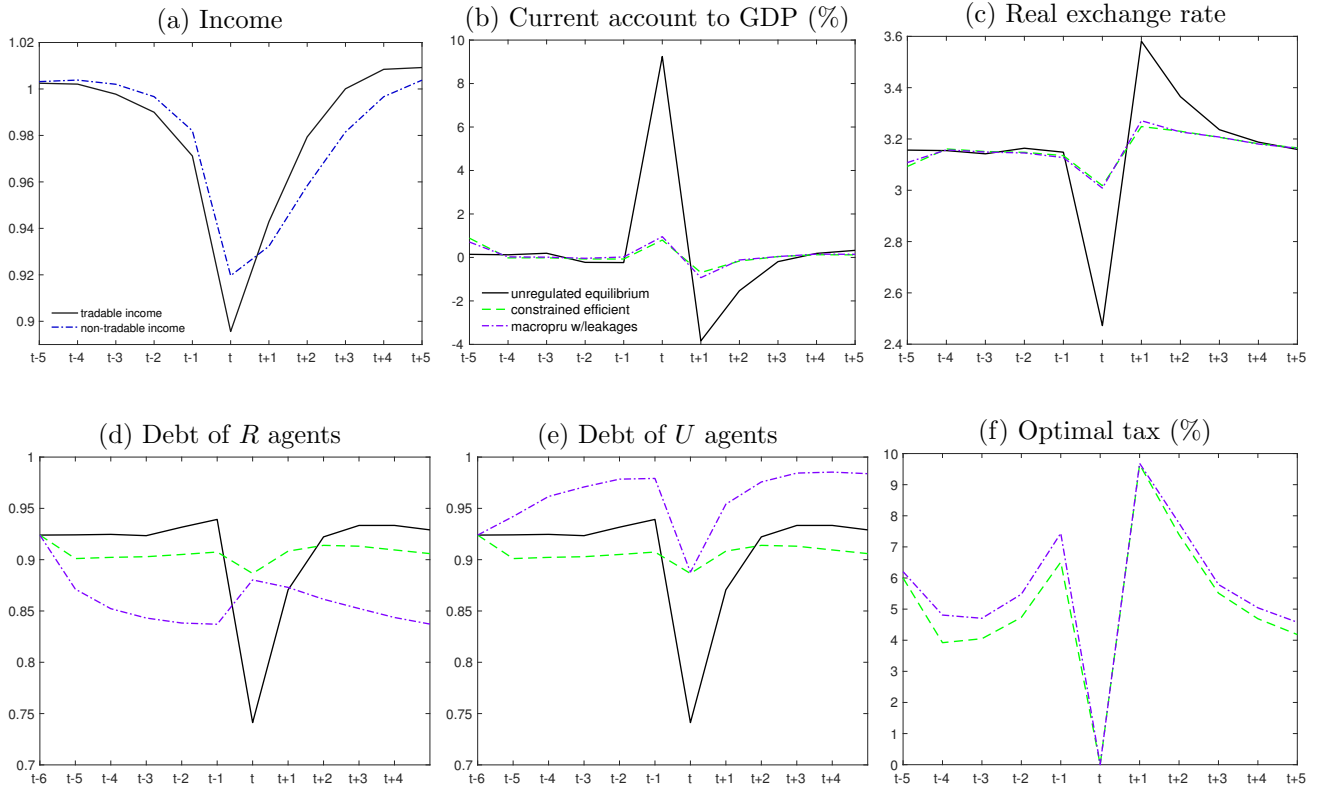


Figure 7: Event analysis.

prudential policy with leakages, dash-dotted line) is much closer to the constrained-efficient equilibrium than to the unregulated equilibrium. This is consistent with the message from Figure 6 that the frequency of crises increases only modestly with γ . That is, overall, neither the frequency nor the severity of financial crises increases substantially with leakages, even when as much as 50% of the economy is left unregulated. However, these aggregate results hide important disparities between the debt dynamics of the regulated and unregulated spheres across the event window. Both debt positions start at exactly the same level (by construction), but unregulated agents start accumulating debt very rapidly (panel e), while regulated agents reduce their indebtedness at a significantly faster pace than in the constrained-efficient equilibrium. Higher taxes than in the constrained-efficient case (panel f), together with stronger precautionary motives due to the spillback effects from unregulated agents' overborrowing, generates this sharp deleveraging by regulated agents in the run-up to the crisis event.

3.7 Welfare effects

Finally, we study the welfare implications of macroprudential policy with leakages, focusing on two key questions: By how much does average welfare fall because of the presence of leakages? How are the welfare benefits of macroprudential policy distributed across regulated and unregulated agents?

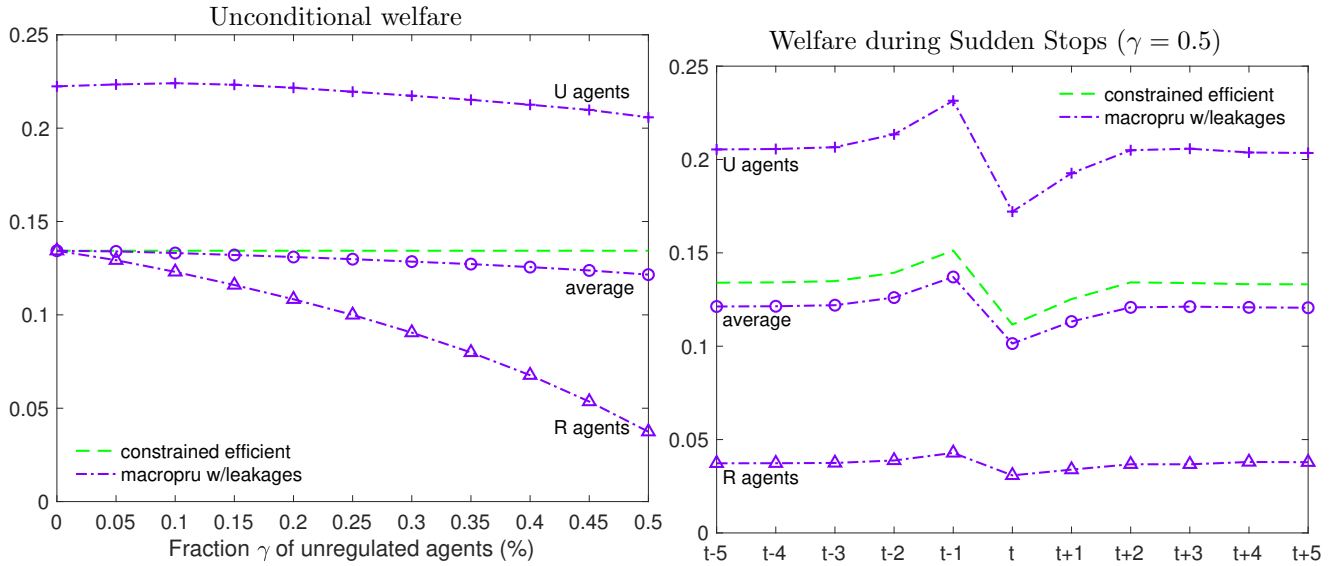


Figure 8: Welfare gains from macroprudential policy in the presence of leakages.

Note: Welfare gains are computed in consumption equivalence terms and expressed in percentages.

Figure 8 displays measures of the welfare effects of macroprudential policy. In the left panel, we report the unconditional welfare gains of moving from the unregulated equilibrium to the regulated equilibrium for different degrees of leakages. We report welfare gains by agent type, as well as average welfare gains.¹⁵ For comparison, we also show the welfare gains of moving from the unregulated equilibrium to the constrained-efficient allocation, which are of course the same for regulated and unregulated agents. A first observation is that the average welfare gains of being in the regulated equilibrium decrease with the size of the unregulated sector. This finding is natural, since with larger γ , the planner directly controls a smaller share of the economy and thus becomes less effective at correcting the overborrowing externality. However, it is also apparent that the decline in the average welfare gains associated with leakages is modest. This suggests that macroprudential policy remains not only effective but also desirable, even with significant leakages.

This figure also reveals interesting insights about the distribution of these welfare effects across the two spheres. First, welfare gains are higher for unregulated agents than for regulated agents. For small values of γ , the welfare gains of macroprudential policy are about twice as large for unregulated agents. The intuition is straightforward: unregulated agents enjoy the same financial stability benefits of macroprudential regulation as regulated agents, but unlike the latter, they do not bear the costs that arise from lower consumption ahead of (potential) future crises. Second, welfare gains for regulated agents fall sharply as γ gets larger. Intuitively,

¹⁵To compute unconditional gains, we first compute, for every possible state (B_U, B_R, y^T, y^N) , the proportional increase in consumption across all possible future histories that would make households indifferent between remaining in the unregulated equilibrium and switching to the regulated equilibrium with leakages. Then, we compute the mean of this variable in the simulations.

larger leakages imply that more unregulated agents overborrow and therefore impose a larger externality on regulated agents, who bear a more concentrated cost regulation.

The right panel of Figure 8 complements this analysis by representing the welfare gains of macroprudential policy with leakages (for $\gamma = 0.5$), but through the event windows of Section 3.6 rather than unconditionally. The results are broadly consistent with the unconditional analysis, with unregulated agents capturing the lion's share of the gains. In addition, it is apparent that the increase in the welfare gains of macroprudential policy in the run-up to a crisis event falls disproportionately on unregulated agents. This strengthens the conclusion that these agents become the main beneficiaries of macroprudential policy when leakages are large.

4 Conclusion

This paper conducted an investigation of macroprudential policies with limited regulation enforcement. We characterize the optimal policy for different degrees of enforcement and examine the extent to which leakages undermine the effectiveness of capital flow management. Our results show that the presence of leakages does not necessarily call for weaker intervention, as commonly argued in policy discussions. Instead, our framework suggests that a stronger intervention may well be needed. Quantitative results show that, thanks to a larger intervention, macroprudential policy remains highly effective at reducing the vulnerability to financial crises.

Even though we conduct our analysis in a small open economy model featuring excessive external borrowing, we think that our insights regarding the two-way interaction between the regulated and unregulated spheres of the economy and the trade-offs that emerge for optimal regulation are likely to apply to a general class of models in which either financial or nominal frictions generate excessive risk taking from a social point of view. Studying the interaction of these frictions and exploring alternative policy responses is an interesting agenda for future research.

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A Formal characterization of continuation equilibria

Lemma 2. For $x \in \{cc, uc, cu, uu\}$, aggregate date 1 consumption in region x is given by $C_1^T(s) = \alpha_{y_1}^x y_1^T(s) + \alpha_U^x B_{U1} + \alpha_R^x B_{R1} + \alpha_{y_2}^x y_2^T$.

Proof. See Appendix B.2. □

Lemma 2 says that, within each region, date 1 aggregate consumption is linear in each of the aggregate state variables $(y_1^T(s), B_{U1}, B_{R1})$. Date 2 aggregate consumption follows from the economy's intertemporal resource constraint (15) and is therefore also linear in $(y_1^T(s), B_{U1}, B_{R1})$. Finally, according to (16), the equilibrium prices $p_1^N(s)$ and $p_2^N(s)$ are linear in $C_1^T(s)$ and $C_2^T(s)$ (respectively) and therefore also linear in $(y_1^T(s), B_{U1}, B_{R1})$. The next lemma characterizes the aggregate consumption solution in more detail.

Lemma 3. The coefficients of the decision rule for $C_1^T(s)$ are such that:

1. $\alpha_{y_1}^x > 0$ and $\alpha_U^x, \alpha_R^x, \alpha_{y_2}^x \geq 0$ with $\alpha_U^x = 0$ (resp. $\alpha_R^x = 0$) iff $\gamma = 0$ (resp. $\gamma = 1$), and $\alpha_{y_2}^x = 0$ iff $x = cc$.
2. if $0 \leq \gamma \leq 0.5$ (resp. $0.5 \leq \gamma \leq 1$), then $\alpha_{y_1}^{uu} \leq \alpha_{y_1}^{cu} \leq \alpha_{y_1}^{uc} \leq \alpha_{y_1}^{cc}$, (resp. $\alpha_{y_1}^{uu} \leq \alpha_{y_1}^{uc} \leq \alpha_{y_1}^{cu} \leq \alpha_{y_1}^{cc}$), with strict inequalities if $0 < \gamma < 0.5$ (resp. $0.5 < \gamma < 1$).
3. $\alpha_U^{uu} \leq \alpha_U^{cu} \leq \alpha_U^{cc}$ and $\alpha_U^{uu} \leq \alpha_U^{uc} \leq \alpha_U^{cc}$, with strict inequalities iff $\gamma > 0$.
4. $\alpha_R^{uu} \leq \alpha_R^{cu} \leq \alpha_R^{cc}$ and $\alpha_R^{uu} \leq \alpha_R^{uc} \leq \alpha_R^{cc}$, with strict inequalities iff $\gamma < 1$.

Proof. See Appendix B.3. □

Part 1. of Lemma 3 establishes that aggregate consumption is increasing in each of the three aggregate state variables $(y_1^T(s), B_{U1}, B_{R1})$, always strictly for $y_1^T(s)$, and strictly for B_{U1} unless $\gamma = 0$ and for B_{R1} unless $\gamma = 1$. Part 2. of the lemma says that aggregate tradable consumption is more sensitive to tradable income in the regions where the credit constraints are binding. Similarly, parts 3. and 4. establish that aggregate tradable consumption is more sensitive to the two sets of agents' wealth positions in the regions where the credit constraints are binding.

We define an individual's credit constraint set as the set of tradable endowment realizations such that her credit constraint is binding:

$$\mathcal{Q}(b_{i1}; B_{U1}, B_{R1}, x) \equiv \{y_1^T(s) \in \mathbb{R}^+ \mid b_{i2}^{unc}(b_{i1}; y_1^T(s), B_{U1}, B_{R1}, x) < b_{i2}^{con}(b_{i1}; y_1^T(s), B_{U1}, B_{R1}, x)\}.$$

where $x \in \{cc, uc, cu, uu\}$ denotes the region in which the economy is and determines the mapping between $(y_1^T(s), B_{U1}, B_{R1})$ and $(p_1^N(s), p_2^N(s))$ relevant to compute b_{i2}^{unc} and b_{i2}^{con} . The four regions

can hence be represented by the following sets:

$$\mathcal{X}^{cc}(B_{U1}, B_{R1}) \equiv \mathcal{Q}(B_{U1}; B_{U1}, B_{R1}, cc) \cap \mathcal{Q}(B_{R1}; B_{U1}, B_{R1}, cc), \quad (\text{A.1})$$

$$\mathcal{X}^{uc}(B_{U1}, B_{R1}) \equiv \mathcal{Q}^c(B_{U1}; B_{U1}, B_{R1}, uc) \cap \mathcal{Q}(B_{R1}; B_{U1}, B_{R1}, uc), \quad (\text{A.2})$$

$$\mathcal{X}^{cu}(B_{U1}, B_{R1}) \equiv \mathcal{Q}(B_{U1}; B_{U1}, B_{R1}, cu) \cap \mathcal{Q}^c(B_{R1}; B_{U1}, B_{R1}, cu), \quad (\text{A.3})$$

$$\mathcal{X}^{uu}(B_{U1}, B_{R1}) \equiv \mathcal{Q}^c(B_{U1}; B_{U1}, B_{R1}, uu) \cap \mathcal{Q}^c(B_{R1}; B_{U1}, B_{R1}, uu). \quad (\text{A.4})$$

Further, we define unions of some of these sets as $\mathcal{X}^{c*} \equiv \mathcal{X}^{cc} \cup \mathcal{X}^{cu}$, $\mathcal{X}^{*c} \equiv \mathcal{X}^{cc} \cup \mathcal{X}^{uc}$, $\mathcal{X}^{u*} \equiv \mathcal{X}^{uu} \cup \mathcal{X}^{uc}$ and $\mathcal{X}^{*u} \equiv \mathcal{X}^{uu} \cup \mathcal{X}^{cu}$. These sets have some intuitive properties, summarized in the following lemmas.

Lemma 4. *There are thresholds a_x and b_x satisfying $0 \leq a_x \leq b_x$ (with $a_x = b_x$ iff $B_{U1} = B_{R1}$) such that $y_1^T(s) \in \mathcal{X}^{cc}$ iff $y_1^T(s) < a_x$, $y_1^T(s) \in \mathcal{X}^{uu}$ iff $y_1^T(s) \geq b_x$, $y_1^T(s) \in \mathcal{X}^{cu}$ iff $a_x \leq y_1^T(s) < b_x$ and $B_{U1} < B_{R1}$; and $y_1^T(s) \in \mathcal{X}^{uc}$ iff $a_x \leq y_1^T(s) < b_x$ and $B_{U1} > B_{R1}$.*

Proof. See Appendix B.4. □

Lemma 4 says that for a given pair (B_{U1}, B_{R1}) , the regions are ordered along the real line, that the poorest type of agents is never unconstrained when the other type is constrained, and that when both types of agents have the same wealth only the symmetric regions cc and uu can arise. It notably implies that \mathcal{X}^{cc} , \mathcal{X}^{uc} , \mathcal{X}^{cu} and \mathcal{X}^{uu} are disjoint, and that their union is \mathbb{R}^+ , meaning that for any triplet $(y_1^T(s), B_{U1}, B_{R1})$ the economy is always in one and only one region. Finally, the next lemma offers comparative statics results.

Lemma 5. *For a given B_{U1} (resp. B_{R1}) and any two B_{R1}, \tilde{B}_{R1} (resp. B_{U1}, \tilde{B}_{U1}) such that $B_{R1} < \tilde{B}_{R1}$ (resp. $B_{U1} < \tilde{B}_{U1}$):*

1. *for $\mathcal{X} = \{\mathcal{X}^{cc}, \mathcal{X}^{c*}, \mathcal{X}^{*c}\}$, if $y_1^T(s) \in \mathcal{X}(B_{U1}, \tilde{B}_{R1})$ (resp. $y_1^T(s) \in \mathcal{X}(\tilde{B}_{U1}, B_{R1})$), then $y_1^T(s) \in \mathcal{X}(B_{U1}, B_{R1})$.*
2. *for $\mathcal{X} = \{\mathcal{X}^{uu}, \mathcal{X}^{u*}, \mathcal{X}^{*u}\}$, if $y_1^T(s) \in \mathcal{X}(B_{U1}, B_{R1})$, then $y_1^T(s) \in \mathcal{X}(B_{U1}, \tilde{B}_{R1})$ (resp. $y_1^T(s) \in \mathcal{X}(\tilde{B}_{U1}, B_{R1})$).*

Proof. See Appendix B.5. □

Part 1. of Lemma 5 says that the region \mathcal{X}^{cc} where both types of agents are credit constrained, and the regions \mathcal{X}^{c*} and \mathcal{X}^{*c} where at least one type of agents is constrained are all shrinking in B_{R1} and B_{U1} . Part 2. says that the region \mathcal{X}^{uu} where both types of agents are unconstrained, and the regions \mathcal{X}^{u*} and \mathcal{X}^{*u} where at least one type of agents is unconstrained are all expanding in B_{R1} and B_{U1} .

A.1 Price and consumption functions

The price functions are related to the aggregate tradable consumption function, given in Lemma 2, by

$$p_1^N(y_1^T(s), B_{U1}, B_{R1}) = \frac{1 - \omega}{\omega} \frac{C_1^T(y_1^T(s), B_{U1}, B_{R1})}{y^N},$$

and

$$p_2^N(y_1^T(s), B_{U1}, B_{R1}) = \frac{(1 - \omega)(1 + r)(1 + r)[\gamma B_{U1} + (1 - \gamma)B_{R1}] + y_1^T(s) + \frac{y_2^T}{1+r} - C_1^T(y_1^T(s), B_{U1}, B_{R1})}{y^N}.$$

The remaining consumption functions are related to these price functions but the relationships depend on which agents are constrained and unconstrained.

When U agents are unconstrained (i.e., in regions uu and uc), their consumption functions are given by

$$\begin{aligned} C_{U1}^T(y_1^T(s), B_{U1}, B_{R1}) &= C_{U2}^T(y_1^T(s), B_{U1}, B_{R1}) = \\ &= \frac{\omega}{1 + \beta} \left[(1 + r)B_{U1} + y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N + \frac{y_2^T + p_2^N(y_1^T(s), B_{U1}, B_{R1})y^N}{1 + r} \right], \end{aligned}$$

and

$$\begin{aligned} C_{Ut}^N(y_1^T(s), B_{U1}, B_{R1}) &= \frac{1 - \omega}{(1 + \beta)p_t^N(y_1^T(s), B_{U1}, B_{R1})} \\ &\times \left[(1 + r)B_{U1} + y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N + \frac{y_2^T + p_2^N(y_1^T(s), B_{U1}, B_{R1})y^N}{1 + r} \right], \end{aligned}$$

for $t = 1, 2$. In contrast, when these agents are constrained (i.e., in regions cu and cc), their consumption functions are given by

$$\begin{aligned} C_{U1}^T(y_1^T(s), B_{U1}, B_{R1}) &= \omega \left\{ (1 + r)B_{U1} + (1 + \kappa) \left[y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N \right] \right\}, \\ C_{U2}^T(y_1^T(s), B_{U1}, B_{R1}) &= \omega \left\{ y_2^T + p_2^N(y_1^T(s), B_{U1}, B_{R1})y^N - \kappa(1 + r) \left[y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N \right] \right\}, \\ C_{U1}^N(y_1^T(s), B_{U1}, B_{R1}) &= (1 - \omega) \frac{(1 + r)B_{U1} + (1 + \kappa) \left[y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N \right]}{p_1^N(y_1^T(s), B_{U1}, B_{R1})}, \\ C_{U2}^N(y_1^T(s), B_{U1}, B_{R1}) &= (1 - \omega) \frac{y_2^T + p_2^N(y_1^T(s), B_{U1}, B_{R1})y^N - \kappa(1 + r) \left[y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N \right]}{p_2^N(y_1^T(s), B_{U1}, B_{R1})}. \end{aligned}$$

Similarly, when R agents are unconstrained (i.e., in regions uu and cu), their consumption functions are given by

$$\begin{aligned} C_{R1}^T(y_1^T(s), B_{U1}, B_{R1}) &= C_{R2}^T(y_1^T(s), B_{U1}, B_{R1}) = \\ &= \frac{\omega}{1 + \beta} \left[(1 + r)B_{R1} + y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N + \frac{y_2^T + p_2^N(y_1^T(s), B_{U1}, B_{R1})y^N}{1 + r} \right], \end{aligned}$$

and

$$C_{Rt}^N(y_1^T(s), B_{U1}, B_{R1}) = \frac{1 - \omega}{(1 + \beta)p_t^N(y_1^T(s), B_{U1}, B_{R1})} \times \left[(1 + r)B_{R1} + y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N + \frac{y_2^T + p_2^N(y_1^T(s), B_{U1}, B_{R1})y^N}{1 + r} \right],$$

for $t = 1, 2$. In contrast, when these agents are constrained (i.e., in regions uc and cc), their consumption functions are given by

$$\begin{aligned} C_{R1}^T(y_1^T(s), B_{U1}, B_{R1}) &= \omega \left\{ (1 + r)B_{R1} + (1 + \kappa) [y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N] \right\}, \\ C_{R2}^T(y_1^T(s), B_{U1}, B_{R1}) &= \omega \left\{ y_2^T + p_2^N(y_1^T(s), B_{U1}, B_{R1})y^N - \kappa(1 + r) [y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N] \right\}, \\ C_{R1}^N(y_1^T(s), B_{U1}, B_{R1}) &= (1 - \omega) \frac{(1 + r)B_{R1} + (1 + \kappa) [y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N]}{p_1^N(y_1^T(s), B_{U1}, B_{R1})}, \\ C_{R2}^N(y_1^T(s), B_{U1}, B_{R1}) &= (1 - \omega) \frac{y_2^T + p_2^N(y_1^T(s), B_{U1}, B_{R1})y^N - \kappa(1 + r) [y_1^T(s) + p_1^N(y_1^T(s), B_{U1}, B_{R1})y^N]}{p_2^N(y_1^T(s), B_{U1}, B_{R1})}. \end{aligned}$$

B Proofs

B.1 Proof of Lemma 1

Note that the private Euler equation (18) is given by $g(b_1^{ue}) = 0$, where

$$\begin{aligned} g(b) &\equiv 1 - \int_{\underline{y}}^{a_x} \frac{\omega}{\alpha_{y1}^{cc} y_1^T(s) + (\alpha_U^{cc} + \alpha_R^{cc})b} dF(y_1^T(s)) - \int_{a_x}^{\infty} \frac{\omega}{\alpha_{y1}^{uu} y_1^T(s) + (\alpha_U^{uu} + \alpha_R^{uu})b + \alpha_{y2}^{uu} y_2^T} dF(y_1^T(s)) \\ &= 1 - \int_{\underline{y}}^{a_x} \frac{\omega (1 - \kappa \frac{1-\omega}{\omega})}{(1 + \kappa) y_1^T(s) + (1 + r)b} dF(y_1^T(s)) - \int_{a_x}^{\infty} \frac{\omega(1 + \beta)}{y_1^T(s) + (1 + r)b + \beta y_2^T} dF(y_1^T(s)) \end{aligned}$$

$g(b)$ is continuous, satisfies $g(b) \rightarrow -\infty$ for some finite b , $\lim_{b \rightarrow \infty} g(b) = 1 > 0$, and $g'(b) > 0$ for the b range over which consumption is positive. It follows that there exists a single b_1^{ue} for which $g(b_1^{ue}) = 0$.

B.2 Proof of Lemma 2

We consider each region in turn.

cc In this case equilibrium is given by the system (12), (13), (14), (15) and (16). This system is block recursive in a linear system in $C_1^T(s)$ and $p_1^N(s)$. Solving this linear system yields the following coefficients for $C_1^T(s)$: $\alpha_{y1}^{cc} = (1 + \kappa) / (1 - \kappa \frac{1-\omega}{\omega})$, $\alpha_U^{cc} = \gamma(1 + r) / (1 - \kappa \frac{1-\omega}{\omega})$, $\alpha_R^{cc} = (1 - \gamma)(1 + r) / (1 - \kappa \frac{1-\omega}{\omega})$ and $\alpha_{y2}^{cc} = 0$.

cu In this case equilibrium is given by the system (12) and (13) for $i = U$, (10) and (11) for $i = R$, (14), (15) and (16). This system is block recursive in a linear system in

$C_1^T(s)$, $C_2^T(s)$, $p_1^N(s)$ and $p_1^N(s)$. Solving this linear system yields the following coefficients for $C_1^T(s)$: $\alpha_{y1}^{cu} = \frac{\gamma(1+\kappa) + \frac{1-\gamma}{\omega} \frac{1}{1+\beta}}{\frac{1-\gamma}{\omega} + \gamma(1-\kappa \frac{1-\omega}{\omega})}$, $\alpha_U^{cu} = \frac{\gamma(1+r) \frac{1}{1+\beta} (\beta + \frac{1-\gamma}{\omega} + \gamma)}{\frac{1-\gamma}{\omega} + \gamma(1-\kappa \frac{1-\omega}{\omega})}$, $\alpha_R^{cu} = \frac{(1-\gamma)(1+r) \frac{1}{1+\beta} (\frac{1-\gamma}{\omega} + \gamma)}{\frac{1-\gamma}{\omega} + \gamma(1-\kappa \frac{1-\omega}{\omega})}$, $\alpha_{y2}^{cu} = \frac{\frac{1-\gamma}{\omega} \frac{\beta}{1+\beta}}{\frac{1-\gamma}{\omega} + \gamma(1-\kappa \frac{1-\omega}{\omega})}$.

uc In this case R agents are constrained, and equilibrium is given by the system (12) and (13) for $i = R$, (11) and (10) for $i = U$, (14), (15) and (16). This system is block recursive in a linear system in $C_1^T(s)$, $C_2^T(s)$, $p_1^N(s)$ and $p_1^N(s)$. Solving this linear system yields the following coefficients for $C_1^T(s)$: $\alpha_{y1}^{uc} = \frac{(1-\gamma)(1+\kappa) + \gamma \frac{1}{\omega} \frac{1}{1+\beta}}{\frac{\gamma}{\omega} + (1-\gamma)(1-\kappa \frac{1-\omega}{\omega})}$, $\alpha_U^{uc} = \frac{\gamma(1+r) \frac{1}{1+\beta} (\frac{\gamma}{\omega} + 1 - \gamma)}{\frac{\gamma}{\omega} + (1-\gamma)(1-\kappa \frac{1-\omega}{\omega})}$, $\alpha_R^{uc} = \frac{(1-\gamma)(1+r) \frac{1}{1+\beta} (\beta + \frac{\gamma}{\omega} + 1 - \gamma)}{\frac{\gamma}{\omega} + (1-\gamma)(1-\kappa \frac{1-\omega}{\omega})}$ and $\alpha_{y2}^{uc} = \frac{\gamma \frac{1}{\omega} \frac{\beta}{1+\beta}}{\frac{\gamma}{\omega} + (1-\gamma)(1-\kappa \frac{1-\omega}{\omega})}$.

uu In this case equilibrium is given by the system (10), (11), (14), (15) and (16). This system is block recursive in a linear system in $C_1^T(s)$ and $C_2^T(s)$. Solving this linear system yields the following coefficients for $C_1^T(s)$: $\alpha_{y1}^{uu} = 1/(1 + \beta)$, $\alpha_U^{uu} = \gamma(1 + r)/(1 + \beta)$, $\alpha_R^{uu} = (1 - \gamma)(1 + r)/(1 + \beta)$ and $\alpha_{y2}^{uu} = \beta/(1 + \beta)$.

B.3 Proof of Lemma 3

The proof of part 1. simply follows from an inspection of the expressions for the coefficients (see proof of Lemma 2 above), noting that Assumption 2 implies $0 < 1 - \kappa \frac{1-\omega}{\omega} < 1$.

The proof of part 2. follows directly from the observations that (1) $\alpha_{y1}^{uu} < 1 < \alpha_{y1}^{cc}$; (2) for $\gamma = 0$, $\alpha_{y1}^{cc} = \alpha_{y1}^{uc}$ and $\alpha_{y1}^{cu} = \alpha_{y1}^{uu}$; (3) for $\gamma = 1$, $\alpha_{y1}^{cc} = \alpha_{y1}^{cu}$ and $\alpha_{y1}^{uc} = \alpha_{y1}^{uu}$; (4) $\partial \alpha_{y1}^{cu} / \partial \gamma > 0$ and $\partial \alpha_{y1}^{uc} / \partial \gamma < 0$; and (5) for $\gamma = 0.5$, $\alpha_{y1}^{cu} = \alpha_{y1}^{uc}$.

For part 3. we observe that if $\gamma = 0$, then $\alpha_U^{uu} = \alpha_U^{cu} = \alpha_U^{uc} = \alpha_U^{cc} = 0$, and that if $\gamma > 0$ assuming that $\alpha_U^{uu} \geq \alpha_U^{cu}$, $\alpha_U^{uu} \geq \alpha_U^{uc}$, $\alpha_U^{cu} \geq \alpha_U^{cc}$ and $\alpha_U^{uc} \geq \alpha_U^{cc}$ individually lead to contradictions.

Similarly, for part 4. we observe that if $\gamma = 1$, then $\alpha_R^{uu} = \alpha_R^{cu} = \alpha_R^{uc} = \alpha_R^{cc} = 0$, and that if $\gamma < 1$ assuming that $\alpha_R^{uu} \geq \alpha_R^{cu}$, $\alpha_R^{uu} \geq \alpha_R^{uc}$, $\alpha_R^{cu} \geq \alpha_R^{cc}$ and $\alpha_R^{uc} \geq \alpha_R^{cc}$ individually leads to contradictions.

B.4 Proof of Lemma 4

Let us define the thresholds $a_x \equiv \min(0, \tilde{a}_x)$,

$$\tilde{a}_x \equiv -\omega \frac{1 - \kappa \frac{1-\omega}{\omega}}{\theta} \max(B_{R1}, B_{U1}) - (1 - \omega)(1 + r) [\gamma B_{U1} + (1 - \gamma) B_{R1}] + \beta \frac{1 - \kappa \frac{1-\omega}{\omega}}{\theta} \bar{y}^T,$$

and

$$b_x \equiv -\frac{1}{\theta} \min(B_{R1}, B_{U1}) - \frac{1-\omega}{\omega} \frac{\kappa(1+r)}{\theta} [\gamma B_{U1} + (1 - \gamma) B_{R1}] + \frac{\beta(1 - \frac{1-\omega}{\omega} \kappa)}{\theta} \bar{y}^T,$$

where

$$\theta \equiv (1 + \kappa) \beta + \kappa \frac{1}{\omega}.$$

It can be easily verified that

1. $b_{i2}^{unc}(B_{i1}; y_1^T(s), B_{U1}, B_{R1}, cc) < b_{i2}^{con}(B_{i1}; y_1^T(s), B_{U1}, B_{R1}, cc)$ for $i = U, R$ is equivalent to $y_1^T(s) < a_x$,
2. $b_{i2}^{unc}(B_{i1}; y_1^T(s), B_{U1}, B_{R1}, uu) \geq b_{i2}^{con}(B_{i1}; y_1^T(s), B_{U1}, B_{R1}, uu)$ for $i = U, R$ is equivalent to $y_1^T(s) \geq b_x$,
3. $b_{U2}^{unc}(B_{U1}; y_1^T(s), B_{U1}, B_{R1}, cu) < b_{U2}^{con}(B_{U1}; y_1^T(s), B_{U1}, B_{R1}, cu)$ and $b_{R2}^{unc}(B_{R1}; y_1^T(s), B_{U1}, B_{R1}, cu) \geq b_{R2}^{con}(B_{R1}; y_1^T(s), B_{U1}, B_{R1}, cu)$ is equivalent to $a_x \leq y_1^T(s) < b_x$ iff $B_{U1} < B_{R1}$, and
4. $b_{U2}^{unc}(B_{U1}; y_1^T(s), B_{U1}, B_{R1}, uc) \geq b_{U2}^{con}(B_{U1}; y_1^T(s), B_{U1}, B_{R1}, uc)$ and $b_{R2}^{unc}(B_{R1}; y_1^T(s), B_{U1}, B_{R1}, uc) < b_{R2}^{con}(B_{R1}; y_1^T(s), B_{U1}, B_{R1}, uc)$ is equivalent to $a_x \leq y_1^T(s) < b_x$ iff $B_{U1} > B_{R1}$.

B.5 Proof of Lemma 5

The proof simply follows from the fact that a_x and b_x are non-increasing in B_{U1} and B_{R1} (see expressions in proof of Lemma 4).

B.6 Proof of Proposition 1

We start by defining

$$h_U(B_{U1}, B_{R1}) \equiv 1 - \mathbb{E}_0 \left[\frac{\omega}{C_{U1}^T(y_1^T(s), B_{U1}, B_{R1})} \right], \quad h_R(B_{U1}, B_{R1}; \tau) \equiv \frac{1}{1 + \tau} - \mathbb{E}_0 \left[\frac{\omega}{C_{R1}^T(y_1^T(s), B_{U1}, B_{R1})} \right] \quad (\text{B.1})$$

and note that the unregulated agents' Euler equation (19) can be written as $h_U(B_{U1}^*, B_{R1}^*) = 0$, while the regulated agents' Euler equation (20) can be written as $h_R(B_{U1}^*, B_{R1}^*; \tau) = 0$.

Let us consider the equilibrium response of U agents. We note that we can write h_U as

$$h_U(B_{U1}, B_{R1}) = 1 - \int_{\underline{y}}^{a_x} \frac{\omega}{C_{U1}^T} dF(y_1^T(s)) - \int_{a_x}^{b_x} \frac{\omega}{C_{U1}^T} dF(y_1^T(s)) - \int_{b_x}^{\infty} \frac{\omega}{C_{U1}^T} dF(y_1^T(s))$$

where the arguments of C_{U1}^T , a_x and b_x are omitted in the interest of space. According to the implicit function theorem we have $dB_{U1}/dB_{R1} = -\frac{\partial h_U/\partial B_{R1}}{\partial h_U/\partial B_{U1}}$, with

$$\begin{aligned} \frac{\partial h_U}{\partial B_{R1}} &= \int_{\underline{y}}^{a_x} \frac{\omega}{(C_{U1}^T)^2} \frac{\partial C_{U1}^T}{\partial B_{R1}} dF(y_1^T(s)) + \int_{a_x}^{b_x} \frac{\omega}{(C_{U1}^T)^2} \frac{\partial C_{U1}^T}{\partial B_{R1}} dF(y_1^T(s)) + \int_{b_x}^{\infty} \frac{\omega}{(C_{U1}^T)^2} \frac{\partial C_{U1}^T}{\partial B_{R1}} dF(y_1^T(s)) \\ \frac{\partial h_U}{\partial B_{U1}} &= \int_{\underline{y}}^{a_x} \frac{\omega}{(C_{U1}^T)^2} \frac{\partial C_{U1}^T}{\partial B_{U1}} dF(y_1^T(s)) + \int_{a_x}^{b_x} \frac{\omega}{(C_{U1}^T)^2} \frac{\partial C_{U1}^T}{\partial B_{U1}} dF(y_1^T(s)) + \int_{b_x}^{\infty} \frac{\omega}{(C_{U1}^T)^2} \frac{\partial C_{U1}^T}{\partial B_{U1}} dF(y_1^T(s)) \end{aligned}$$

where we used the fact that terms containing derivatives of a_x and b_x drop out due to the continuity of C_{U1}^T across regions.¹⁶ The derivatives in the various regions are given by

$$\begin{aligned}
cc : \quad \frac{\partial C_{U1}^T}{\partial B_{R1}} &= \frac{\omega(1+r)(1+\kappa)\frac{1-\omega}{\omega}(1-\gamma)}{1-\kappa\frac{1-\omega}{\omega}}; & \frac{\partial C_{U1}^T}{\partial B_{U1}} &= \omega(1+r) \left[1 + \frac{(1+\kappa)\frac{1-\omega}{\omega}\gamma}{1-\kappa\frac{1-\omega}{\omega}} \right]. \\
cu : \quad \frac{\partial C_{U1}^T}{\partial B_{R1}} &= \frac{\omega(1+r)(1+\kappa)\frac{1-\omega}{\omega}(1-\gamma) \left(\frac{1-\gamma}{\omega} + \gamma \right)}{(1+\beta) \left[\frac{1-\gamma}{\omega} + \gamma \left(1 - \kappa\frac{1-\omega}{\omega} \right) \right]}; \\
& \frac{\partial C_{U1}^T}{\partial B_{U1}} &= \omega(1+r) \left[1 + \frac{(1+\kappa)\frac{1-\omega}{\omega}\gamma \left(\beta + \frac{1-\gamma}{\omega} + \gamma \right)}{(1+\beta) \left[\frac{1-\gamma}{\omega} + \gamma \left(1 - \kappa\frac{1-\omega}{\omega} \right) \right]} \right] \\
uc : \quad \frac{\partial C_{U1}^T}{\partial B_{R1}} &= \frac{\omega(1+r)\frac{1-\omega}{\omega}(1-\gamma)}{1+\beta}; & \frac{\partial C_{U1}^T}{\partial B_{U1}} &= \frac{\omega(1+r) \left[1 + \frac{1-\omega}{\omega}\gamma \right]}{1+\beta} \\
uu : \quad \frac{\partial C_{U1}^T}{\partial B_{R1}} &= \frac{\omega(1+r)\frac{1-\omega}{\omega}(1-\gamma)}{(1+\beta)}; & \frac{\partial C_{U1}^T}{\partial B_{U1}} &= \frac{\omega(1+r) \left[1 + \frac{1-\omega}{\omega}\gamma \right]}{(1+\beta)}
\end{aligned}$$

Therefore, in every region the term $\frac{\partial C_{U1}^T}{\partial B_{R1}}$ is non-negative (and strictly positive whenever $\gamma < 1$) and the term $\frac{\partial C_{U1}^T}{\partial B_{U1}}$ is strictly positive. It follows that for a given B_R , $\partial h_U / \partial B_{U1} > 0$ in the range of B_U for which C_{U1}^T is always positive. The equilibrium response of U agents to B_R is therefore unique, and can be written as $B_U = \phi_U(B_R)$. Further, in the range of B_U and B_R for which C_{U1}^T is always positive, we have $\partial h_U / \partial B_{R1} \geq 0$, with strict inequality whenever $\gamma < 1$. It follows that $\phi'_U(B_R) \leq 0$, with strict inequality whenever $\gamma < 1$.

For the equilibrium response of R agents, the proof is analogous and involves the derivatives of C_{R1}^T in the four regions. The equilibrium response of R agents to B_U is unique and decreasing, strictly whenever $\gamma > 0$.

B.7 Proof of Proposition 2

The proof relies on the relationship between the slopes ϕ'_U and $[\partial \phi_R / \partial B_U]^{-1}$ at the unregulated equilibrium debt choices, as well as on the sign of the partial derivative $\partial \phi'_R(B_U, \tau_R) / \partial \tau_R$.

We have $\phi'_U = -\frac{\partial h_U / \partial B_{R1}}{\partial h_U / \partial B_{U1}}$ and $\partial \phi_R / \partial B_U = -\frac{\partial h_R / \partial B_{U1}}{\partial h_R / \partial B_{R1}}$. At the unregulated equilibrium we have $\tau_R = 0$, $B_{U1}^* = B_{R1}^* = B_1^{ue}$, and therefore $a_x = b_x$ and $C_{U1}^T = C_{R1}^T = C_1^T$ for any state at date 1. Defining $\eta_{cc} = \int_{\underline{y}}^{\alpha_x} \frac{1}{(C_1^T)^2} dF(y_1^T(s))$ and $\eta_{uu} = \int_{\alpha_x}^{\infty} \frac{1}{(C_1^T)^2} dF(y_1^T(s))$, we have

$$\phi'_U(B_1^{ue}) = -\frac{\eta_{cc}(1-\gamma)\frac{(1+\kappa)\frac{1-\omega}{\omega}}{1-\kappa\frac{1-\omega}{\omega}} + \eta_{uu}(1-\gamma)\frac{1}{1+\beta}\frac{1-\omega}{\omega}}{\eta_{cc} \left[1 + \gamma \frac{(1+\kappa)\frac{1-\omega}{\omega}}{1-\kappa\frac{1-\omega}{\omega}} \right] + \eta_{uu} \frac{1}{1+\beta} \left[1 + \gamma \frac{1-\omega}{\omega} \right]} \quad (\text{B.2})$$

¹⁶Note that if $B_{U1} < B_{R1}$ the relevant intermediate region between a_x and b_x is $x = cu$, while if $B_{U1} > B_{R1}$ the relevant region is $x = uc$. If $B_{U1} = B_{R1}$ then $a_x = b_x$ so this intermediate region drops out.

and

$$[\partial\phi_R(B_1^{ue}, 0)/\partial B_U]^{-1} = -\frac{\eta_{cc} \left[1 + (1 - \gamma) \frac{(1+\kappa) \frac{1-\omega}{\omega}}{1-\kappa \frac{1-\omega}{\omega}}\right] + \eta_{uu} \frac{1}{1+\beta} \left[1 + (1 - \gamma) \frac{1-\omega}{\omega}\right]}{\eta_{cc} \gamma \frac{(1+\kappa) \frac{1-\omega}{\omega}}{1-\kappa \frac{1-\omega}{\omega}} + \eta_{uu} \gamma \frac{1}{1+\beta} \frac{1-\omega}{\omega}} \quad (\text{B.3})$$

For any value of γ , the numerator in (B.2) is smaller than the one in (B.3), and the denominator in (B.2) is larger than the one in (B.3). It follows that $|\phi'_U(B_1^{ue})| < |[\partial\phi_R(B_1^{ue}, 0)/\partial B_U]^{-1}|$ and therefore $\phi'_U(B_1^{ue}) > [\partial\phi_R(B_1^{ue}, 0)/\partial B_U]^{-1}$.

Furthermore, we have

$$\frac{\partial\phi'_R(B_U, \tau_R)}{\partial\tau_R} = -\frac{\partial h_R/\partial\tau_R}{\partial h_R/\partial B_R} = -\frac{-1/(1 + \tau_R)^2}{\partial h_R/\partial B_R} > 0$$

since $\partial h_R/\partial B_R > 0$.

In the (B_{U1}, B_{R1}) space, the curve $\phi_R(B_U, \tau_R)$ crosses the curve $\phi_U(B_R)$ from above at $(B_{U1}, B_{R1}) = (B_1^{ue}, B_1^{ue})$, and it shifts to the right as τ_R rises. Hence, in the neighborhood of (B_1^{ue}, B_1^{ue}) , a rise in τ_R results in a downward movement of (B_{U1}^*, B_{R1}^*) along the downward sloping $\phi_U(B_R)$ curve. It follows that for a small τ_R , B_{U1}^* is decreasing in τ_R and B_{R1}^* is increasing in τ_R .

The result on debt capacity follows from the sign of the derivative $\frac{dp_1^N}{dB_{R1}} \equiv \frac{\partial p_1^N}{\partial B_{R1}} + \frac{\partial p_1^N}{\partial B_{U1}} \frac{\partial B_{U1}}{\partial B_{R1}}$. Evaluated at $(B_{U1}, B_{R1}) = (B_1^{ue}, B_1^{ue})$, this derivative is given by

$$\begin{aligned} \frac{dp_1^N}{dB_{R1}} &= \frac{1 - \omega}{\omega} \left[\frac{(1 - \gamma)(1 + r)}{1 - \kappa \frac{1-\omega}{\omega}} + \frac{\gamma(1 + r)}{1 - \kappa \frac{1-\omega}{\omega}} \phi'_U(B_1^{ue}) \right] \\ &= \frac{1 - \omega}{\omega} \frac{(1 - \gamma)(1 + r)}{1 - \kappa \frac{1-\omega}{\omega}} \left[\frac{\eta_{cc} + \eta_{uu} \frac{1}{1+\beta}}{\eta_{cc} \left[1 + \gamma \frac{(1+\kappa) \frac{1-\omega}{\omega}}{1-\kappa \frac{1-\omega}{\omega}}\right] + \eta_{uu} \frac{1}{1+\beta} \left[1 + \gamma \frac{1-\omega}{\omega}\right]} \right] > 0 \end{aligned}$$

in region cc , and by

$$\begin{aligned} \frac{dp_1^N}{dB_{R1}} &= \frac{1 - \omega}{\omega} \left[\frac{(1 - \gamma)(1 + r)}{1 + \beta} + \frac{\gamma(1 + r)}{1 + \beta} \phi'_U(B_1^{ue}) \right] \\ &= \frac{1 - \omega}{\omega} \frac{(1 - \gamma)(1 + r)}{1 + \beta} \left[\frac{\eta_{cc} + \eta_{uu} \frac{1}{1+\beta}}{\eta_{cc} \left[1 + \gamma \frac{(1+\kappa) \frac{1-\omega}{\omega}}{1-\kappa \frac{1-\omega}{\omega}}\right] + \eta_{uu} \frac{1}{1+\beta} \left[1 + \gamma \frac{1-\omega}{\omega}\right]} \right] > 0 \end{aligned}$$

in region uu . Since cc and uu are the only two relevant regions when debt choices are symmetric, it follows that $\frac{dp_1^N}{dB_{R1}} > 0$.

B.8 Proof of Proposition 3

The proof relies on the result that starting from the unregulated equilibrium, a movement along the best response of unregulated agents associated with a small tax τ_R leads to welfare gains for

agents of both types. To establish this result, we start by observing that

$$V_{i1}(y_1^T, B_{U1}, B_{R1}) = \max_{C_{i1}^T, C_{i1}^N, C_{i2}^T, C_{i2}^N} \sum_{t=1}^2 \beta^{t-1} [\omega \ln(C_{it}^T) + (1-\omega) \ln(C_{it}^N)]$$

subject to

$$\sum_{t=1}^2 \frac{1}{(1+r)^{t-1}} [(C_{it}^T - y_t^T) + p_t^N(y_1^T, B_{U1}, B_{R1}) (C_{it}^N - y_t^N)] = (1+r)B_{i1}$$

and

$$(1+r)B_{i1} + (C_{i1}^T - y_1^T) + p_1^N(y_1^T, B_{U1}, B_{R1}) (C_{i1}^N - y_1^N) + \kappa [y_1^T(s) + p_1^N(y_1^T, B_{U1}, B_{R1})y_1^N] \geq 0.$$

Therefore, using the Envelope theorem, we have

$$\frac{\partial V_{i1}}{\partial B_{i1}} = (1+r) \frac{\omega}{C_{i1}^T} + \omega \left(\frac{1}{C_{i1}^T} - \frac{1}{C_{i2}^T} \right) \kappa y^N \frac{\partial p_1^N}{\partial B_{i1}} + \sum_{t=1}^2 \frac{1}{(1+r)^{t-1}} \frac{\omega}{C_{i1}^T} (y^N - C_{i1}^N) \frac{\partial p_t^N}{\partial B_{i1}} \quad (\text{B.4})$$

and

$$\frac{\partial V_{i1}}{\partial B_{j1}} = \omega \left(\frac{1}{C_{i1}^T} - \frac{1}{C_{i2}^T} \right) \kappa y^N \frac{\partial p_1^N}{\partial B_{j1}} + \sum_{t=1}^2 \frac{1}{(1+r)^{t-1}} \frac{\omega}{C_{i1}^T} (y^N - C_{i1}^N) \frac{\partial p_t^N}{\partial B_{j1}} \quad (\text{B.5})$$

for $j \neq i$.

Next, the date 0 welfare of an agent of type i associated with aggregate debt choices (B_{U1}, B_{R1}) can be written as

$$W_{i0}(B_{U1}, B_{R1}) = -B_{i1} + \beta \mathbb{E}_0 V_{i1}(y_1^T, B_{U1}, B_{R1}).$$

Thus, the variation in welfare of an agent of type i caused by imposing a small tax on regulated agents is proportional to

$$\frac{dW_{i0}}{dB_{R1}} = \frac{\partial W_{i0}}{\partial B_{R1}} + \frac{\partial W_{i0}}{\partial B_{U1}} \frac{\partial B_{U1}}{\partial B_{R1}}.$$

For unregulated agents, this variation is given by

$$\frac{dW_{U0}}{dB_{R1}} = \beta \mathbb{E}_0 \frac{\partial V_{U1}}{\partial B_{R1}} + \left(-1 + \beta \mathbb{E}_0 \frac{\partial V_{U1}}{\partial B_{U1}} \right) \frac{\partial B_{U1}}{\partial B_{R1}}.$$

Using (B.4) and (B.5) and evaluating the expression at the unregulated equilibrium, the variation reduces to

$$\frac{dW_{U0}}{dB_{R1}} = \beta \mathbb{E}_0 \left[\omega \left(\frac{1}{C_{U1}^T} - \frac{1}{C_{U2}^T} \right) \kappa y^N \frac{dp_1^N}{dB_{R1}} \right].$$

Under the premise that borrowing constraints bind with positive probability in the unregulated equilibrium, the term in round brackets is positive in some states of nature (and zero in the others). Furthermore, Proposition 2 established that $\frac{dp_1^N}{dB_{R1}} > 0$. It follows that $\frac{dW_{U0}}{dB_{R1}} > 0$.

For regulated agent, the welfare variation is given by

$$\frac{dW_{R0}}{dB_{R1}} = -1 + \beta \mathbb{E}_0 \frac{\partial V_{R1}}{\partial B_{R1}} + \beta \mathbb{E}_0 \frac{\partial V_{R1}}{\partial B_{U1}} \frac{\partial B_{U1}}{\partial B_{R1}}.$$

Using (B.4) and (B.5) and evaluating the expression at the unregulated equilibrium, the variation reduces to

$$\frac{dW_{R0}}{dB_{R1}} = \beta \mathbb{E}_0 \left[\omega \left(\frac{1}{C_{R1}^T} - \frac{1}{C_{R2}^T} \right) \kappa y^N \frac{dp_1^N}{dB_{R1}} \right].$$

Again, under the premise that borrowing constraints bind with positive probability in the unregulated equilibrium, the term in round brackets is positive in some states of nature (and zero in the others), while Proposition 2 established that $\frac{dp_1^N}{dB_{R1}} > 0$. It follows that $\frac{dW_{R0}}{dB_{R1}} > 0$.

B.9 Proof of Proposition 4

The proof of the “if” part is by construction. Assume that the tax is zero. $\tau_R = 0$ implies that $(B_{U1}, B_{R1}) = (B_1^{ue}, B_1^{ue})$, which implies symmetric allocations in all states of the world at date 1 and 2: $C_{Ut}^T = C_{Rt}^T$ and $C_{Ut}^N = C_{Rt}^N$ for $t = 1, 2$. The optimal tax expression (24) then implies

$$\tau_R = \frac{\beta \mathbb{E}_0 \left[\left(\mu_{R1} + \frac{\gamma}{1-\gamma} \mu_{U1} \right) \kappa \bar{y}^N \frac{dp_1^N}{dB_{R1}} \right]}{\mathbb{E}_0 \left[\frac{\omega}{C_{R1}^T} \right]} = 0$$

since $\mu_{R1} = \mu_{U1} = 0$ in all states of the world. $\tau_R = 0$ is therefore indeed optimal.

The proof of the “only if” part is by contradiction. Assume that the tax is zero and that credit constraint binds, i.e. $\mu_{R1} > 0$ and/or $\mu_{U1} > 0$, in some states of the world in the decentralized equilibrium. $\tau_R = 0$ implies that $(B_{U1}, B_{R1}) = (B_1^{ue}, B_1^{ue})$, which induces symmetric allocations in all states of the world at date 1 and 2: $C_{Ut}^T = C_{Rt}^T$ and $C_{Ut}^N = C_{Rt}^N$ for $t = 1, 2$. The optimal tax expression (24) then implies

$$\tau_R = \frac{\beta \mathbb{E}_0 \left[\left(\mu_{R1} + \frac{\gamma}{1-\gamma} \mu_{U1} \right) \kappa \bar{y}^N \frac{dp_1^N}{dB_{R1}} \right]}{\mathbb{E}_0 \left[\frac{\omega}{C_{R1}^T} \right]} \quad (\text{B.6})$$

The proof of Proposition 2 established that $\frac{dp_1^N}{dB_{R1}} > 0$ necessarily hold at the unregulated equilibrium debt choices. Thus, (B.6) implies $\tau_R > 0$, a contradiction.

C Recursive Competitive Equilibrium

We present the optimization problem of a representative agent in recursive form. The aggregate state vector of the economy is $X = \{B_U, B_R, y^T, y^N\}$. The state variables for a type i agent’s problem is the individual state b_i and the aggregate states X . Agents need to forecast the future

price of nontradables. To this end, they need to forecast future aggregate bond holdings. We denote by $\Gamma_i(\cdot)$ the forecast of aggregate bond holdings for the set of type i agents for every current aggregate state X , i.e., $B'_i = \Gamma_i(X)$. Combining first-order conditions equilibrium conditions c^T, c^N , budget constraints and market clearing, the forecast price function for nontradables can be expressed as

$$p^N(X) = \frac{1 - \omega}{\omega} \left(\frac{y_t^N + [\gamma B_U + (1 - \gamma) B_R](1 + r) - [\gamma \Gamma_U(X) + (1 - \gamma) \Gamma_R(X)]}{y^N} \right)^{\eta+1}. \quad (\text{B.7})$$

The problem of a type i agent can then be written as:

$$V(b_i, X) = \max_{b'_i, c_i^T, c_i^N} u(c(c_i^T, c_i^N)) + \beta \mathbb{E}V(b'_i, X') \quad (\text{B.8})$$

subject to

$$\begin{aligned} b'_i + p^N(X)c_i^N + c_i^T &= b_i(1 + r) + p^N(X)y^N + y^T \\ b'_i &\geq -\kappa(p^N(X)y^N + y^T) \\ B'_j &= \Gamma_j(X) \quad \text{for } j = \{U, R\} \end{aligned}$$

The solution to this problem yields decision rules for individual bond holdings $\hat{b}(b_i, X)$, tradable goods consumption $\hat{c}^T(b_i, X)$ and nontradable goods consumption $\hat{c}^N(b_i, X)$. The decision rule for bond holdings induces actual laws of motion for aggregate bonds, given by $\hat{b}(B_i, X)$.

Definition of unregulated Equilibrium. A *recursive unregulated equilibrium* is defined by a pricing function $p^N(X)$, perceived laws of motions $\Gamma_i(X)$ for $i \in \{U, R\}$, and decision rules $\hat{b}(b_i, X)$, $\hat{c}^T(b_i, X)$, $\hat{c}^N(b_i, X)$ with associated value function $V(b_i, X)$ such that:

1. Agents' optimization: $\{\hat{b}(b_i, X), \hat{c}^T(b_i, X), \hat{c}^N(b_i, X)\}$ and $V(b_i, X)$ solve the agent's i recursive optimization problem for $i \in \{U, R\}$, taking as given $p^N(X)$ and $\Gamma_i(X)$ for $i = \{U, R\}$.
2. Consistency: the perceived laws of motion for aggregate bonds are consistent with the actual laws of motion: $\Gamma_i(X) = \hat{b}(B_i, X)$ for $i = \{U, R\}$.
3. Market clearing:

$$\gamma \hat{c}^N(B_U, X) + (1 - \gamma) \hat{c}^N(B_R, X) = y^N$$

and

$$\gamma [\Gamma_U(X) + \hat{c}^T(B_U, X) - B_U(1 + r)] + (1 - \gamma) [\Gamma_R(X) + \hat{c}^T(B_R, X) - B_R(1 + r)] = y^T.$$

D Numerical solution

To solve for the regulated equilibrium when $\gamma > 0$, we adopt a nested fixed point algorithm similar to those used in studies of Markov perfect equilibria (e.g., Bianchi and Mendoza, 2018, Klein et al., 2005, Klein et al., 2008). Given future policies, we solve for policy functions and value functions using value function iteration in an inner loop. In the outer loop, we update future policies with the solution of the Bellman equation from the inner loop. The algorithm follows these steps:

1. Generate a discrete grid for the bond position of regulated agents $G_R = \{b_1, b_2, \dots, b_M\}$, the bond position of unregulated agents $G_U = \{b_1, b_2, \dots, b_M\}$ and the shocks $G_Y = \{s_1, s_2, \dots, s_N\}$ and choose an interpolation scheme for evaluating the functions outside the bond grids. We use 100 points for each bond grid and interpolate using piecewise linear approximation.
2. Guess policy functions $\mathcal{U}'_U, \mathcal{P}$ at step $K \forall b_R \in G_R, \forall b_U \in G_U$ and $\forall y \in G_Y$. We use as initial policies the policies of the unregulated equilibrium.
3. For given $\mathcal{U}'_U, \mathcal{P}$, solve for the value function and policy functions associated with the following Bellman equation:

$$V(X) = \max_{\{c_i^T, c_i^N, b'_i\}_{i \in \{U, R\}}, \mu, p^N} \gamma u(c_U^T, c_U^N) + (1 - \gamma)u(c_R^T, c_R^N) + \beta \mathbb{E}V(X') \quad (\text{B.9})$$

subject to

$$b'_R = (1 + r)b_R + y^T - c_R^T + p^N(y^N - c_R^N) \quad (\text{B.10})$$

$$b'_U = (1 + r)b_U + y^T - c_U^T + p^N(y^N - c_U^N) \quad (\text{B.11})$$

$$p^N = \frac{1 - \omega}{\omega} \left(\frac{\gamma c_U^T + (1 - \gamma)c_R^T}{y^N} \right)^{\eta+1} \quad (\text{B.12})$$

$$c_R^N = \frac{c_R^T}{\gamma c_U^T + (1 - \gamma)c_R^T} y^N \quad (\text{B.13})$$

$$c_U^N = \frac{c_U^T}{\gamma c_U^T + (1 - \gamma)c_R^T} y^N \quad (\text{B.14})$$

$$b'_R \geq -\kappa(\mathcal{P}(X)y^N + y^T) \quad (\text{B.15})$$

$$b'_U \geq -\kappa(\mathcal{P}(X)y^N + y^T) \quad (\text{B.16})$$

$$\mu = u_T(c_U^T, c_U^N) - \beta(1 + r)\mathbb{E}\mathcal{U}'_U(X') \quad (\text{B.17})$$

$$\mu \times [b'_U + \kappa(\mathcal{P}(X)y^N + y^T)] = 0 \quad (\text{B.18})$$

$$\mu \geq 0. \quad (\text{B.19})$$

The Bellman equation is solved using value function iteration. In each state, the maxi-

mization is performed using discrete search for b'_R on the G_R grid. For a given b'_R , the value of all remaining variables (including b'_U), and therefore of the objective, follow from (B.10)-(B.19).

4. Denote by $\sigma^{\mathcal{U}}$ and $\sigma^{\mathcal{P}}$ the policy functions that solve the recursive problem in step 3. Compute the sup distance between \mathcal{U}_U and $\sigma^{\mathcal{U}}$, as well as between \mathcal{P} and $\sigma^{\mathcal{P}}$. If the sup distance is higher than 1.0e-5, update \mathcal{U}_U and \mathcal{P} and go back to step 2.