Partial Bank Runs and Optimal Default [*]

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Abstract

We investigate the efficiency of banks' defaults using a dynamic general equilibrium model. The model features a continuum of identical banks that trade assets in a competitive market and may default due to fundamentals or self-fulfilling runs. For intermediate debt values, the competitive equilibrium must be of a mixed type, where a fraction of banks default and a fraction of banks repay. In the absence of self-fulfilling runs, we show that there are too few defaults relative to the social optimal, while there are too many defaults in the presence of self-fulfilling runs.

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1 Introduction

Bank failures are a common feature of financial crises, and concerns over their adverse consequences often elicit policy interventions aimed at reducing the incidence of bankruptcies. In this paper, we ask: Is the incidence of bank failures necessarily inefficient?

To answer this question, we present a dynamic general equilibrium model that allows for the possibility that banks default due to fundamentals or self-fulfilling runs. We show that bankruptcies may have a cleansing effect on the banking system in the case where crises are triggered by fundamentals. We argue that when more banks fail, this keeps asset prices low and helps those banks that survive a crisis because they are net buyers of assets. Because these banks have a high marginal value of funds, this general equilibrium effect leads to welfare gains at the margin.

The theory builds on the model we developed in Amador and Bianchi (2024) where we study the effects of credit easing. In the model, banks have limited commitment, and their default decisions depend on asset prices, which are determined endogenously in the model. When investors panic and refuse to roll over deposits from a bank, the bank must raise liquidity by either cutting equity payouts or selling its assets holdings. If the liquidity problem is severe, it becomes optimal for the bank to default, making the run a self-fulfilling equilibrium outcome. The bank may also default because of fundamentals. This occurs when the bank finds it optimal to default regardless of whether investors are willing to roll over.

The normative analysis we consider in this paper examines how the planner would choose the fraction of banks that default (and how this compares to the competitive equilibrium). To isolate the critical inefficiency at play, we consider a version of the model with identical banks. At the same time, we allow banks to face the possibility of runs in all future periods.[1]

Our first set of results is a complete analytical characterization of banks' policies in partial equilibrium for given asset prices. Using closed-form solutions for the bank's value function, we derive an endogenous dynamic borrowing limit. On the surface, the borrowing limit resembles Kiyotaki and Moore (1997), but here the fraction that banks, in effect, pledge as collateral is endogenous. In particular, the borrowing limit at any point in time is increasing in future borrowing limits. As future borrowing limits are relaxed, the continuation value for a repayment bank increases, making it more attractive for a bank to repay today. Although the feedback between future and current borrowing limits may open the possibility for multiple equilibria, we show that the limit is unique. Our characterization shows that while there are two solutions for the borrowing limit, only one solution satisfies the No-Ponzi game and is unstable. We then introduce

[1] The model in Amador and Bianchi (2024) feature banks that face an idiosyncratic shock in period 0 and runs are assumed to be possible in the initial period.

the possibility of bank runs and show how this tightens borrowing limits by making repayment less attractive in the future.

Our second set of results concerns the general equilibrium characterization. Our work generalizes the analysis in Kehoe and Levine (1993) and Alvarez and Jermann (2000) for the possibility of initial defaults in equilibrium.[2] We characterize the two possible stationary equilibrium outcomes, default and repayment, and show that the stationary outcome is uniquely determined in the absence of runs. Transitional dynamics can then be separated into three regions. When aggregate leverage is low, the economy converges to a stationary equilibrium in which all banks always repay. In this region, asset prices are high, reflecting banks' high productivity and the fact that more capital helps relax the borrowing limit. When aggregate leverage is high, all banks default, and asset prices are depressed as capital is priced by low-productive agents.

For intermediate values of aggregate leverage, there is no degenerate equilibrium. In this case, the competitive equilibrium features mixed strategies where a fraction of banks repay and a fraction of banks default. We refer to this equilibrium as *partial runs*. Furthermore, we show that the price of capital converges to the stationary equilibrium price under repayment. This occurs through a gradual reallocation of capital from defaulting to repaying banks where average productivity goes up over time. Interestingly, because the fraction of repaying banks is an endogenous equilibrium outcome, a policy targeted to support a subset of defaulting banks is ineffective.

Turning to the normative analysis, we consider the welfare implications of a policy that directly controls the share of defaulting banks. Perhaps surprisingly, in the absence of runs, increasing the share of defaulting banks may increase banks' welfare. This result stems from a general equilibrium effect operating through asset prices that effectively redistributes resources within the banking sector. When banks demand more capital, they raise the price of capital, hurting those banks that are net buyers of assets. Because repaying banks are net buyers of capital and have higher marginal utility of consumption, increasing the share of defaulting banks reduces the market clearing price of capital and increases banks' overall welfare. When the economy is subject to runs, lowering the share of defaulting banks relative to the competitive equilibrium may increase banks' welfare. This is so because defaults are driven by a coordination problem: banks are defaulting even though they would strictly prefer to repay if they had access to credit. In this case, a policy of reducing defaults would generate a Pareto improvement, as it also benefits lenders; a result that highlights the inefficiency of the competitive equilibrium.

Related literature. Our paper is related to a vast literature on the role of financial frictions for macroeconomic fluctuations, following the work of Bernanke and Gertler (1989) and Kiyotaki and

[2] Like in their work, defaults do not arise on the equilibrium path for $t > 0$.

Moore (1997).[3] Our environment without runs is related to the literature on investment under limited commitment, and in particular, the papers of Thomas and Worrall (1994) and Alburquerque and Hopenhayn (2004) in partial equilibrium and the general equilibrium analysis of Kehoe and Levine (1993), Cooley, Marimon and Quadrini (2004), and Alvarez and Jermann (2000). Different from this literature, we incorporate the possibility of self-fulfilling runs, following the formulation of Cole and Kehoe (2000) and show how the expectation of runs can tighten further borrowing limits. A key contribution of our paper is to analyze the existence of mixing equilibrium where a fraction of banks default in the initial period and to analyze the normative implications.

The version of our model with bank runs connects with a large literature, starting with Bryant (1980) and Diamond and Dybvig (1983).[4] As mentioned above, we build on Amador and Bianchi (2024). In that model, however, mixing equilibrium does not emerge because banks face idiosyncratic shocks in period 0, and in addition, runs may only occur in the initial period.

Our normative analysis connects with a literature on the efficiency properties of competitive equilibrium with imperfect financial markets. This literature has examined how various reallocation of assets may be desirable due to pecuniary externalities.[5] Our paper contributes to this literature by analyzing the efficiency properties of private defaults on debt obligations.

Outline. Section 2 presents the environment and analyzes the model without runs. Section 3 introduces bank runs. Section 4 conducts the normative analysis. Section 5 concludes. All proofs are in the Appendix.

2 Model

Time is discrete and infinite, $t \in \{0, 1, 2, \ldots\}$. There is a single final consumption good and no aggregate shocks. The economy is populated by a continuum of financial institutions, which we refer to as banks, and creditors, both of measure one. In what follows, we use small variables to denote individual variables and capital letters to denote aggregate variables.

[3] A few examples include Gertler and Kiyotaki (2010), Mendoza (2010), Jermann and Quadrini (2012), He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014), and Bianchi and Mendoza (2018).

[4] A few examples include Cooper and Ross, 1998, Ennis and Keister, 2009, Dávila and Goldstein, 2020, and Keister and Narasiman (2016), Allen and Gale (2009), Allen and Gale (2000) and Uhlig (2010), Gertler and Kiyotaki (2015), Gertler, Kiyotaki and Prestipino (2020).

[5] One branch of this literature has focused on limited spanning (e.g., Hart, 1975, Geanakoplos and Polemarchakis, 1985, Lorenzoni, 2008) and a second branch has focused on price-dependent borrowing constraints (Kehoe and Levine, 1993, Alvarez and Jermann, 2000, Bianchi, 2011). See also Dávila and Korinek (2018) for an overview of the distinctive implications of these two externalities.

Technology. Production of the final consumption good uses capital, k, as a single input. We assume that banks have direct access to the production technology, in line with the most recent strand of macro-finance models. A bank with k units of capital produces $y = zk$ units of consumption. Capital does not depreciate, and it is in fixed supply.

Preferences. Banks' preferences over a sequence of dividend payments $\{c_t\}$ are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta \in (0,1)$ and $u = \log$.[6]

Banks' creditors are risk neutral and discount payoffs at a rate R. Given these assumptions, the risk-free rate will be constant and equal to R.

2.1 Banks' Problem and Borrowing Limits

We describe now the problem of an individual bank in partial equilibrium. Banks choose bond issuances, investment, dividend payments and whether to repay the existing deposits. In this section, we consider banks' defaults only due to fundamentals. We introduce bank runs in Section 3.

Banks issue one-period bonds that promise a payment of R next period. A bank starts a period t with k units of capital and b units of maturing bonds, and decides whether to repay or to default. If the bank chooses to repay, it produces using a linear technology with productivity $\bar{z}$, and chooses its new holding of capital for the next period $k' \geq 0$, the new amount of bonds to issue, b', and how many dividends to pay, c. The bank faces a price schedule $q_t(b', k')$ for its bonds, that depends on its individual choices for new bonds and capital, as well as other aggregate variables which we summarized in t. These variables determine the incentives to default in the next period and hence alter the price at which creditors are willing to lend.

In case of repayment, the bank's budget constraint is

$$c = (\bar{z} + p_t)k - Rb + q_t(b', k')b' - p_t k', \tag{1}$$

where p_t denotes the price of capital in period t.

If the bank chooses to default, it is permanently excluded from bond markets and can only

[6] As is standard in the literature, the curvature in the utility function over dividends (or equity payouts) captures the fact that issuing equity is costly.

invest in capital.[7] In addition, the bank's productivity is reduced to to $\underline{z} < \bar{z}$. In the case of default, the budget constraint is

$$c = (\underline{z} + p_t)k - p_t k'. \tag{2}$$

The problem of the individual bank that is in good credit standing solves

$$V_t(b,k) = \max\{V_t^R(b,k), V_t^D(k)\}, \tag{3}$$

where the value of default is given by

$$V_t^D(k) = \max_{k' \geq 0, c} \; \log(c) + \beta V_{t+1}^D(k'), \tag{4}$$

$$\text{subject to}$$

$$c = (\underline{z} + p_t)k - p_t k',$$

and the value of repayment is

$$V_t^R(b,k) = \max_{k' \geq 0, b', c} \; \log(c) + \beta V_{t+1}(b', k') \tag{5}$$

$$\text{subject to}$$

$$c = (\bar{z} + p_t)k - Rb + q_t(b', k')b' - p_t k'.$$

We will also make sure that the bond price schedule q_t is consistent with a No-Ponzi condition for the bank, which we discuss below.

Using $d = 0$ to represent a repayment decision, and $d = 1$, a default, we have that the optimal default rule is

$$d_t(b,k) = \begin{cases} 1 & \text{if } V_t^R(b,k) < V_t^D(k), \\ 0 & \text{if } V_t^R(b,k) > V_t^D(k), \\ 0 & \text{if } V_t^R(b,k) = V_t^D(k) \text{ for } t > 0, \end{cases} \tag{6}$$

where we assume, without loss of generality, that the bank repays if indifferent for $t > 0$. However, we do not restrict the default policy in period 0 when the bank is indifferent. That is, we allow for the bank to default in period 0 even if indifferent. This flexibility is important for the existence of a general equilibrium, as we will see below.[8]

[7] The restriction that the bank cannot hold bonds after default is without loss of generality if the rate of return to capital in equilibrium for a bank that has defaulted is higher than R. This is guaranteed in the general equilibrium in which all banks default discussed in Section 2.3.

[8] The reason why assuming that the bank pays if indifferent for $t > 0$ is without loss of generality is as follows. If banks were to randomize when indifferent for $t > 0$ (with some arbitrary probability), it would be optimal for the bank to choose a level of debt ϵ below the indifferent point and borrow at a price of 1.

Noting that $V_t^R(b,k)$ is strictly decreasing in b, we have that for every k, there exists a borrowing limit $\bar{b}_t$ such that $V_t^D(k) > V_t^R(b,k)$ if and only if $b > \bar{b}_t$. This means that the optimal default rule can be expressed with a debt threshold (which we assume to be finite for every $k \geq 0$):

$$d_t(b,k) = \begin{cases} 1 & \text{if } b > \bar{b}_t(k), \\ 0 & \text{if } b \leq \bar{b}_t(k), \end{cases}$$

for $t > 0$. It thus follows that the equilibrium price schedule for bonds is going to be of the form

$$q_t(b',k') = \begin{cases} 0 & \text{if } b' > \bar{b}_{t+1}(k'), \\ 1 & \text{if } b' \leq \bar{b}_{t+1}(k'), \end{cases}$$

for $t > 0$. That is, creditors lend at a zero price when they expect a default and lend at a price of 1 when they expect repayment. Note that because banks will never issue bonds at a zero price, default can only occur in equilibrium in the initial period.

Given a sequence of the price of capital, we define the return to capital when the bank repays as

$$R_{t+1}^k \equiv \frac{\bar{z} + p_{t+1}}{p_t},$$

for all t. Similarly, we define the return to capital when the bank defaults as

$$R_{t+1}^D \equiv \frac{\underline{z} + p_{t+1}}{p_t},$$

for all t. Note that our assumptions about a productivity loss after default imply that $R_{t+1}^k > R_{t+1}^D$.

The value of default. For a given sequence of prices, $\{p_t\}_{t=0}^{\infty}$, we can solve for the value of default, exploiting the log-utility and the linearity of production. We introduce the following condition (which guarantees the boundedness of the value of default).

Condition 1. *The sequence of (strictly positive) prices $\{p_t\}_{t=0}^{\infty}$ is such that*

$$\lim_{t\to\infty} \beta^t \log\left(R_{t+1}^D\right) = 0.$$

We have the following result:

Lemma 1 (The value of default)**.** *Suppose that Condition 1 holds. Then the value of default, $V_t^D(k)$*

in period t is finite and such that

$$V_t^D(k) = A + \frac{1}{1-\beta}\log((\underline{z}+p_t)k) + \frac{\beta}{1-\beta}\sum_{\tau\geq t}\beta^{\tau-t}\log\left(R_{\tau+1}^D\right), \tag{7}$$

with

$$A \equiv \frac{1}{1-\beta}\left[\log(1-\beta) + \frac{\beta}{1-\beta}\log(\beta)\right].$$

Proof. In Appendix A.1. □

Condition 1 is a sufficient condition for the value function of default to be finite, and it requires that the returns of capital do not grow at a faster rate than the discount factor.

The value function is log-linear in wealth and the discounted future returns on capital. The associated policy function for capital, $\mathcal{K}_{t+1}^D(k)$, and dividend payout, $C_{t+1}^D(k)$, are given by,

$$\mathcal{K}_{t+1}^D(k) = \beta\frac{(\underline{z}+p_t)k}{p_t},$$

$$C_t^D(k) = (1-\beta)\left(\underline{z}+p_t\right)k.$$

Because of log preferences, the optimal policy is independent of future returns. In particular, the bank consumes a fraction $(1-\beta)$ of its net worth, which equals $(\underline{z}+p_t)k$, and invest the remaining amount in capital. Under this investment policy, the evolution of net worth is given by

$$n' = \beta R_{t+1}^D n.$$

The value of repayment. Given a sequence of prices $\{p_t\}_{t=0}^{\infty}$, we can express the value function of repayment as follows:

Lemma 2 (The repayment problem)**.** *There exists a function $\hat{V}_t^R$ such that $\hat{V}_t^R(n) = V_t(b,k)$, where*

$$n = (\bar{z}+p_t)k - Rb,$$

and $\hat{V}^R$ solves

$$\hat{V}_t^R(n) = \max_{k'\geq 0, b', c}\ \log(c) + \beta\hat{V}_{t+1}^R(n'), \tag{8}$$

$$\text{subject to}$$

$$c = n + b' - p_t k',$$

$$n' = (\bar{z}+p_{t+1})k' - Rb',$$

$$b' \leq \bar{b}_{t+1}(k').$$

Proof. In Appendix A.2. □

Note that relative to Problem (5), we have used that we can summarize the individual state variables in net worth $n = (\bar{z} + p_t)k - Rb$.

We refer to this problem as partial equilibrium, since it takes as given the path of prices $\{p_t\}_{t=0}^{\infty}$. However, the problem does incorporate the equilibrium bond price function for an individual bank. That is, the last constraint of Problem (8) uses that a bank never issues bonds at a zero price, and effectively the equilibrium bond price schedule imposes a borrowing limit.[9] This borrowing limit takes for now the role of a No-Ponzi condition (in that it helps guarantee that the budget set is not unbounded), but we will refine this later on.

We now guess that the value function under repayment (if finite) will be log-linear in net worth. In particular, we guess that $\hat{V}_{t+1}^{R}(n) = \frac{1}{1-\beta}\log(n) + \text{constant}$. Given that $\bar{b}_{t+1}(k)$ is determined by the equality of default and repayment values, $V_{t+1}^{D}(k') = \hat{V}_{t+1}^{R}(n')$, at $n' = (\bar{z} + p_{t+1})k' - R\bar{b}_{t+1}(k')$, it follows then that there exists a γ_t such that $\bar{b}_{t+1}(k') = \gamma_t p_{t+1} k'$.[10] Thus, the bank will be subject to a borrowing constraint

$$b_{t+1} \leq \gamma_t p_{t+1} k_{t+1},$$

where γ_t is an endogenous object representing an individual bank's ability to leverage at time t, which it will itself be affected by the sequence of prices of capital.

Note that in Problem (8), it is always feasible for a repaying bank to choose $b' = 0$, as long as $(\bar{z} + p)k - Rb = n \geq 0$. Hence, the borrowing limit $\bar{b}_t$ cannot be negative; that is, $\gamma_t \geq 0$ for all t.

The next lemma characterizes the demand for capital:

Lemma 3. *Consider a repaying bank in period t with strictly positive net-worth.*

(i) If $\gamma_t p_{t+1} \geq p_t$, and $R_{t+1}^{k} > R$, the bank's demand for capital in period t and its value function are infinite.

(ii) If $\gamma_t p_{t+1} < p_t$ and $R_{t+1}^{k} > R$, the bank's demand for capital in period t is finite and is such that the borrowing constraint binds.

(iii) If $R_{t+1}^{k} < R$, the bank's demand for capital in period t is 0.

Proof. In Appendix A.3. □

[9] These constraints are the equivalent of the "not too tight" solvency constraints introduced by Alvarez and Jermann (2000), although an important difference with their environment is the presence of capital, production, and default cost in ours. In this environment without risk, the borrowing constraints also coincide with the endogenous borrowing constraints used by Zhang (1997).

[10] Note that in effect, we have scaled the value of the borrowing limit by p_{t+1}. This is without loss of generality and will become useful in what follows.

The first result of this lemma concerns the case where the return to capital is higher than R, and the bank ability to leverage is sufficiently large. When $\gamma_t p_{t+1} > p_t$, a repaying bank can invest any amount simply by borrowing and investing. This result follows because when a bank borrows one unit and purchases capital, the borrowing capacity increases by $\gamma_t p_{t+1}/p_t$. When this ratio is larger than 1, it is feasible for the bank to purchase an unlimited amount of capital while still paying positive dividends. To the extent that the return on capital exceeds the return on bonds, the bank will find it optimal to invest an infinite amount, and the value of the bank will be unbounded.[11] When $\gamma_t p_{t+1} = p_t$, a similar result applies, but only if networth of the bank is strictly positive (which guarantees a strictly positive dividend).

Part (ii) of the lemma covers the case where the return to capital is strictly higher than R, but $\gamma_t p_{t+1} < p_t$. In that case, the borrowing limit binds.

The last result of the lemma is for $R^k_{t+1} < R$. In this case, investing in capital is dominated in rate of return by holding the bond.

Let us define the levered return on equity as

$$R^e_{t+1} \equiv R^k_{t+1} + (R^k_{t+1} - R)\frac{\gamma_t p_{t+1}}{p_t - \gamma_t p_{t+1}}, \tag{9}$$

which corresponds to the sum of the return on capital plus the excess return (of capital over bonds) times a leverage factor.[12] We need to impose as well a condition on R^e_t to guarantee the boundedness of the value of repayment for an individual bank, similar to Condition 1 for the case of a defaulting bank. Taking the above lemmas together, and anticipating the general equilibrium, we restrict attention to sequences of prices and borrowing limits that satisfy the following.

Condition 2. *The sequences of prices $\{p_t\}_{t=0}^{\infty}$ and $\{\gamma_t\}_{t=0}^{\infty}$ are such that*

(i) $R^k_{t+1} \geq R$ *for all* t,

(ii) $\gamma_t p_{t+1} < p_t$ *for every* t *such that* $R^k_{t+1} > R$,

(iii) $\lim_{t\to\infty} \beta^t \log\left(R^e_t\right) = 0$.

Note that part (iii) of this condition implies Condition 1 as $R^e_t \geq R^k_t > R^D_t > 0$.

We can now solve for the value function of repayment (confirming that it is log-linear in net worth) as well as characterizing the associated policy functions.

[11] Thus, the result also holds if the bank has negative equity at the beginning of the period and $\gamma_t p_{t+1} > p_t$. Negative equity is not sufficient to prevent a bank from operating. The condition is also necessary: that is, for a bank to be able to operate with negative networth, it must be the case that $\gamma_t p_{t+1} > p_t$.

[12] If the bank has an additional unit of net worth and buys capital, it can borrow an additional $\gamma_t p_{t+1}/p_t$ by pledging the capital as collateral. In turn, the increase in borrowing allows for further purchases of capital. If $\gamma_t p_{t+1} < p_t$, the amount it can borrow is $\gamma_t p_{t+1}/(p_t - \gamma_t p_{t+1})$. The return per unit of leverage is $R^k_t - R$, thus leading to (9).

Lemma 4 (The value of repayment). *Consider a sequence of (strictly positive) prices, $\{p_t\}_{t=0}^{\infty}$ and (non-negative) borrowing limits, $\{\gamma_t\}_{t=0}^{\infty}$, that satisfy Condition 2. Then, the value of repayment $\hat{V}_t^R(n)$ and associated policy functions in period t for $n > 0$ are such that:*

(i) Value function:

$$\hat{V}_t^R(n) = A + \frac{1}{1-\beta}\log(n) + \frac{\beta}{1-\beta}\sum_{\tau \geq t}\beta^{\tau-t}\log(R_{\tau+1}^e), \tag{10}$$

with A as in Lemma 1.

(ii) Policy functions:

$$C_t^R(n) = (1-\beta)n,$$

for all $t \geq 0$ and where $\mathcal{K}_{t+1}^R(n)$ and $\mathcal{B}_{t+1}^R(n)$, satisfy

$$p_t\mathcal{K}_{t+1}^R(n) - \mathcal{B}_{t+1}^R(n) = \beta n, \qquad \mathcal{B}_{t+1}^R(n) \leq \gamma_t p_{t+1}\mathcal{K}_{t+1}^R(n), \qquad \mathcal{K}_{t+1}^R(n) \geq 0$$

for all $t \geq 0$. And

$$\mathcal{K}_{t+1}^R(n) = \frac{\beta n}{p_t - \gamma_t p_{t+1}}, \qquad \mathcal{B}_{t+1}^R(n) = \gamma_t p_{t+1}\left(\frac{\beta n}{p_t - \gamma_t p_{t+1}}\right)$$

for all $t \geq 0$ such that $R_{t+1}^k > R$.

Proof. In Appendix A.4. □

Thus, under repayment, the problem also features a value function that is log-linear in net worth, confirming our previous guess. The value is also log-linear in the discounted future returns of the portfolio. In addition, the dividend payout is given by a fraction of the net worth. Note that the problem is quite similar to the default one, except that we use the net worth (which requires subtracting the beginning of period debt) from the gross return on investment. On the other hand, the problem under repayment features higher returns, both because there is a higher productivity level, and thus $R^k > R^D$, and because the bank can lever up if $R^k > R$ and $\gamma > 0$.

Regarding the portfolio, the solution distinguishes between the case in which $R^k = R$ and $R^k > R$. If the return on capital is equal to the return on debt, the bank is indifferent between bonds and capital and chooses any portfolio as long as it is consistent with the dividend policy and the leverage constraint. If the return on capital exceeds the one on debt, the bank borrows to the limit.

Using the results of Lemma 4, we can express the evolution of net worth as

$$n' = \beta R^e_{t+1} n$$

for all $t \geq 0$. Hence, next-period net worth is given by the amount of net worth that is not consumed, βn, times the return on equity. Note that this is the same law of motion for equity under default, but it uses the rate of return on equity R^e under repayment rather than the return on capital R^D under default.

Default decision. Having characterized the values of repayment and default, we can now examine the default decision. The following proposition establishes the value of the leverage threshold, γ, at which the bank is indifferent between repaying and defaulting.

Proposition 1 (Default decision)**.** *Consider a sequence of (strictly positive) prices, $\{p_t\}_{t=0}^{\infty}$, and a sequence of (non-negative) borrowing limits, $\{\gamma_t\}_{t=0}^{\infty}$ that satisfy Condition 2. Then, the value of γ_t that makes a bank indifferent between repayment and default at $t+1$ is such that*

$$\frac{\bar{z} + p_{t+1}(1 - \gamma_t R)}{\underline{z} + p_{t+1}} = \left(1 - \gamma_{t+1}\frac{p_{t+2}}{p_{t+1}}\right)^{\beta} \qquad \textit{for all } \; t \geq 0. \tag{G}$$

Proof. In Appendix A.5. □

The sequence for default thresholds $\{\gamma_t\}$ depends on preference, productivity parameters, and the sequence for $\{p_t\}$.[13] One can see, in particular, that a higher γ_{t+1} in the future implies a higher γ_t today. Because a higher γ_{t+1} increases the continuation value of repayment, this also makes the bank more willing to repay today.

The above suggests that there could be potentially many sequences of borrowing limits, $\{\gamma_t\}$, that would be consistent with a partial equilibrium given a sequence of capital prices. For an equilibrium to be consistent with creditors' optimality, we also require a no-Ponzi game condition. That is,

$$\lim_{t\to\infty} R^{-t} b_t \leq 0$$

where $\{b_t\}$ is a feasible sequence of debt issuances. This condition says that creditors in the limit, do not provide new loans to finance the repayment of old ones. Using that $b_{t+1} \leq \gamma_t p_{t+1} \frac{\beta n_t}{p_t - \gamma_t p_{t+1}}$, together with the evolution of net worth, we impose the no-Ponzi condition as an additional restriction to the sequence of $\{\gamma_t\}$:

[13] Note that once γ_0 has been determined, equation (G) determines a γ_{-1} that can be used to characterize the default decision in the first period.

Condition 3. *The sequence of prices $\{p_t\}_{t=0}^{\infty}$ and $\{\gamma_t\}_{t=0}^{\infty}$ are such that*

$$\lim_{t\to\infty}\left[\prod_{\tau=0}^{t}\left(\frac{\beta R_t^e}{R}\right)\right]\left(\frac{\beta\gamma_t p_{t+1}}{p_t-\gamma_t p_{t+1}}\right)\leq 0$$

As we will see below, this condition uniquely pins down the sequence of $\{\gamma_t\}$ given a sequence of prices $\{p_t\}$. Effectively, if p_t converges and γ_t remains bounded away from 0, the condition above imposes that the growth rate of net worth cannot be higher than the interest rate R in the limit.

With this, we can characterize the sequence of γ_t that are consistent with bank's and creditor's optimality conditions, given a sequence of prices:

Definition 1. Given a sequence of (strictly positive) prices $\{p_t\}_{t=0}^{\infty}$, we say a sequence of (non-negative) borrowing limits $\{\gamma_t\}_{t=0}^{\infty}$ is *equilibrium-consistent* if Conditions 2 and 3 hold and equation (G) is satisfied for all $t \geq 0$.

Note that if we have found a sequence of (non-negative) borrowing limits, $\{\gamma_t\}$, that satisfy the above definition, we can construct the evolution of net worth, debt, and capital holdings consistent with a bank's optimality condition by using the results of Lemmas 4 for a given initial net worth, $n_0 > 0$.

A useful case is the one where the sequence of prices $\{p_t\}$ is constant. We proceed to analyze this case next.

2.2 Equilibrium-Consistent Borrowing Limits under a Constant Price

We now focus on the case in which the price of capital is constant at some level $p > 0$. In that case, the return to capital, $R^k = (\bar{z}+p)/p$ is constant as well. Note that Condition 1 is immediately satisfied. We also require that $R^k \geq R$ to satisfy the first inequality in Condition 2. Note that this last condition imposes an upper bound on p (as $R > 1$).

Let us focus on the equation described in Proposition 1, equation (G). For the constant price case, the equation is:

$$\gamma_{t+1} = 1-\left(\frac{R^k/R-\gamma_t}{R^D/R}\right)^{\frac{1}{\beta}} \equiv H(\gamma_t) \tag{11}$$

where R^D is the return to capital under default with a constant price (that is, $R^D = (\underline{z}+p)/p$).

The function H describes the value of the value of next-period borrowing limit, γ_{t+1}, that is consistent with a current borrowing limit, γ_t, when the price of capital is constant. So for any initial guess of γ_0, we can use this difference equation $\gamma_{t+1} = H(\gamma_t)$ to trace out all of the subsequent values for γ_t. Notice that if the sequence for $\{\gamma_t\}$ converges to a constant value, this value must be a fixed point of H.

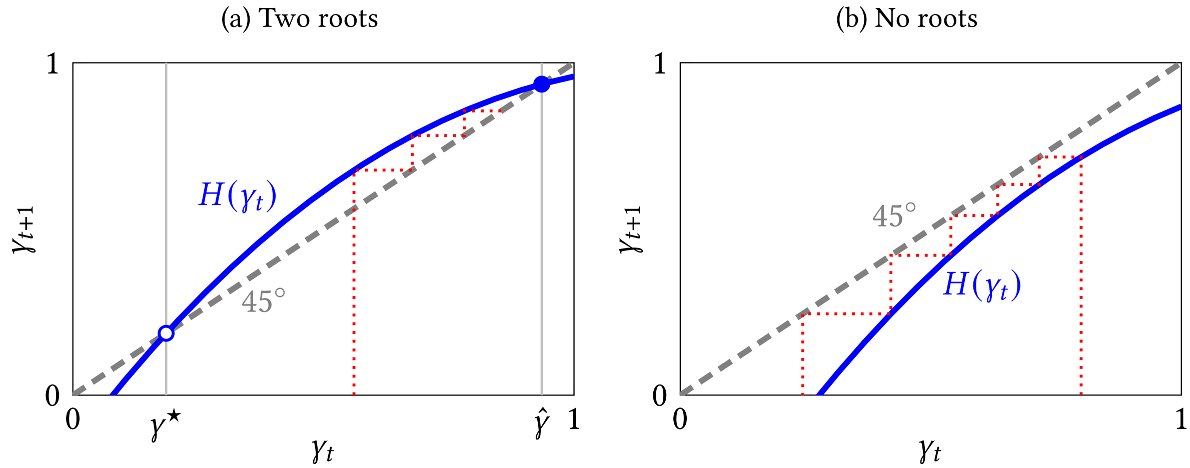

Figure 1: Borrowing Limits with a Constant Price

Notes: The solid curved line represents the H function. The dashed line is the 45° line. Panel (a) shows the case with two roots to $\gamma = H(\gamma)$. The point $\gamma^\star$ represents the valid stationary solution. The point $\hat{\gamma}$ is the stationary root that violates the no-Ponzi condition. Any sequence $\{\gamma_t\}$ that starts from a value different from $\gamma^\star$ eventually either induces a negative γ_t or the sequence approaches $\hat{\gamma}$. Panel (b) shows the case with no roots.

Fixed points of *H*. The function H is well defined, continuous, differentiable and strictly concave in $[0, 1]$. In addition, $H(0) < 0$ and $H(1) \leq 1$. Using that $R^k > R^D$, the following lemma characterizes the fixed points of H:[14]

Lemma 5. *The following holds for H:*

(i) If $\beta R^k/R < \beta + (1-\beta)\left(\beta R^D/R\right)^{\frac{1}{1-\beta}}$ *and* $\beta R^D/R < 1$ *then there are two solutions to* $\gamma = H(\gamma)$ *for* $\gamma \in [0, 1]$.

(ii) If $\beta R^k/R = \beta + (1-\beta)\left(\beta R^D/R\right)^{\frac{1}{1-\beta}}$ *and* $\beta R^D/R < 1$ *then there is only one solution to* $\gamma = H(\gamma)$ *for* $\gamma \in [0, 1]$.

(iii) If $\beta R^k/R > \beta + (1-\beta)\left(\beta R^D/R\right)^{\frac{1}{1-\beta}}$ *or* $\beta R^D/R \geq 1$, *then* $H(\gamma) < \gamma$ *for all* $\gamma \in [0, 1]$.

Proof. In Appendix A.6 □

[14]In the case in which $R^k = R^D$ (which we do not consider) so that there is no productivity loss after a default, it can be shown that $\gamma = 0$ (that is, no borrowing is possible) is a solution to $\gamma = H(\gamma)$. The result for this case can be seen as a corollary of a well-known result for sovereign debt (Bulow and Rogoff, 1989).

Lemma 5 states that equation (11) could have two fixed points, only one fixed point or no fixed points. The precise solution depends on the relative return of capital under repayment and default. When the return of capital under default is not too low compared to the return of capital under repayment, there are two fixed points. On the other hand, when the return of capital under default is very low, there is no fixed point solution. Finally, at an exact intermediate threshold, there is one fixed point solution to H.

Let us provide some intuition for these results. First, why could there be two stationary solutions for γ? This feature arises because of the complementarity of borrowing constraints over time. When the bank faces a loose borrowing constraint at $t+1$ (i.e., a high γ_{t+1}), this implies that tomorrow a repaying bank can attain high profits by leveraging up. This in turn implies that the borrowing constraints at time t is relatively loose (i.e., a high γ_t). This complementarity opens the door to multiple fixed points. The lemma shows, in particular, that there are at most two fixed points. As we argue next, however, only the smallest fixed point is equilibrium-consistent, as the largest fixed point violates the no-Ponzi condition. At the largest fixed point, the bank never repays any interest from the debt to creditors, violating Condition 3.[15]

Lemma 5 also points to the possibility that equation (11) admits no fixed-point, which implies that there is no constant value of γ that makes banks indifferent between repaying and defaulting for given prices. In this case, there exists no finite borrowing limit for the bank.

Figure 1 illustrates the results of Lemma 5. Panel (a) considers case (i): a parameter configuration such that there are two fixed points of H. Panel (b) considers case (iii) a parameter configuration such are no fixed points of H.

Solution for γ_t and comparative statics. Before characterizing the solution for γ_t, it is helpful first to consider the largest stationary value of γ that would be consistent with the no-Ponzi condition. We denote this value by γ^{NP}. Note that in a stationary environment, the no-Ponzi condition will be violated for any $\gamma < 1$ if $\beta R^e \geq R$.[16] Using this result, we obtain that

$$\gamma^{NP} \equiv \frac{R - \beta R^k}{R(1-\beta)} \tag{12}$$

Note that if $\beta R^k > R$, then any stationary solution for $\gamma > 0$ violates Condition 3. The reason is that, even with no access to borrowing, a bank's net worth necessarily grows faster than the discount rate R.

In this stationary environment, we next argue that γ_t must be equal to the smallest fixed point

[15] Notice that even though the bank is running a Ponzi scheme, the bank's value remains finite.

[16] This follows because under this condition, debt would grow at a faster rate than the interest rate, violating the transversality condition for creditors.

at all times, a value we denote by $\gamma^\star$. To understand the argument, consider first the possibility that $\gamma_t < \gamma^\star$. In this case, the borrowing constraint is relatively tight today and equation (11) tells us that to justify a "low" γ_t today, one needs an expectation of an even lower γ_{t+} tomorrow. In other words, to keep banks indifferent from repaying and defaulting at relatively low leverage levels, it must be that borrowing constraints will keep tightening in the future. However, iterating forward on equation (11) will lead eventually to a negative value of γ (a result displayed in panel (b) of the figure), a violation of the equilibrium requirement that the borrowing limit must be non-negative.[17] This rules out $\gamma_t < \gamma^\star$.

Consider now the possibility that $\gamma_t > \gamma^\star$. Tracing again the dynamics using equation (11), we can see in panel (a) of Figure 1 that γ_t converges to the largest fixed-point of H. This fixed-point turns out to be inconsistent with the no-Ponzi game condition (that is, for this case γ converges to a value larger than γ^{NP}, hence ruling out the possibility that $\gamma_t > \gamma^\star$.

We summarize these results in the following lemma:

Lemma 6 (Borrowing limits under a constant price)**.** *Consider a constant price of capital $p > 0$ such that $R^k \geq R$.*

(i) If $\beta R^k/R < \beta + (1-\beta)\left(\beta R^D/R\right)^{\frac{1}{1-\beta}}$ and $\beta R^D/R < 1$. Then, the unique equilibrium-consistent sequence of borrowing limits $\{\gamma_t\}_{t=0}^{\infty}$ is such that $\gamma_t = \gamma^\star$ for all t where $\gamma^\star$ is the smallest solution to $\gamma = H(\gamma)$ for $\gamma \in [0,1]$.

(ii) Otherwise, there exists no equilibrium-consistent sequence of finite borrowing limits.

Proof. In Appendix A.7. □

The lemma above also shows that when $\beta R^k/R = \beta + (1-\beta)\left(\beta R^D/R\right)^{\frac{1}{1-\beta}}$, then there is also no equilibrium-consistent sequence of borrowing limits even though there is a fixed-point to $\gamma = H(\gamma)$ in $[0,\bar{\gamma}]$. The reason is that, in this case, such a γ corresponds exactly to the case in which banks' net worth (and as a result, its debt level) grows at rate R, implying a violation of the no-Ponzi condition.

We proceed now to describe some comparative statics:

Corollary 1 (Comparative Statics)**.** *Consider a constant price of capital $p > 0$ such that $R^k > R$ and $\beta R^k/R < \beta + (1-\beta)\left(\beta R^D/R\right)^{\frac{1}{1-\beta}}$. Then $\gamma^\star$ as defined in part (i) of Lemma 6 is strictly decreasing in $\underline{z}$, R, and p, and strictly increasing in $\bar{z}$ and β.*

Proof. In Appendix A.8. □

[17] Recall that a bank with positive networth can always choose not to issue debt while investing in capital, and thus a negative borrowing limit is inconsistent with an equilibrium .

This corollary provides comparative statics with respect to key parameters. Regarding the productivity parameters, we have that $\gamma^\star$ is increasing in $\bar{z}$ and decreasing in $\underline{z}$. These results are intuitive: the value of repayment for the bank is increasing in $\bar{z}$ and independent of $\underline{z}$, while the value of default is increasing in $\underline{z}$ and independent of $\bar{z}$. Graphically, this result can be seen in panel (a) of Figure 1 by noting that an increase in $\bar{z}$, or a decrease in $\underline{z}$, shifts down the H curve and moves its first intersection with the 45 degree line (which represents the equilibrium-consistent borrowing limit) towards a higher value.

In addition, we have that $\gamma^\star$ is decreasing in R. A bank in default does not save/borrow in bonds, and hence the value of default is independent of R. On the other hand, the value of repayment is decreasing in R because banks are borrowers. As a result, the borrowing constraint becomes less tight with a lower R. Moreover, a higher β also raises $\gamma^\star$ because a higher patience increases the present discount value of the productivity losses upon default.

The effects of the price of capital on $\gamma^\star$ are more subtle because the price of capital affects both the value of repayment and default. In both cases, a decline inthe steady state's price of capital increases the steady state return of investment. What is important to recognize, however, is that a bank in repayment can lever up and have a larger increase in the return on the overall portfolio compared to a bank in default. As a result, an increase in the return on capital increases more the value of repayment than the value of default. Hence, the partial equilibrium default threshold $\gamma^\star$ is decreasing in the price of capital.

2.3 General Equilibrium

In the previous section, we described the problem of an individual bank in partial equilibrium for a given price of capital $\{p_t\}$. As we just saw, the price of capital is key to determine banks' policies and the borrowing limit faced by banks. In this section, we close the model by showing how the market price of capital is determined and characterize equilibria.

Market clearing requires that the aggregate demand for capital from banks equals $\overline{K}$. Because all banks are assumed to be identical at the beginning of time, and there is a measure one of banks, each bank owns $k_0 = \overline{K}$ units of the capital stock and has a debt level $b_0 = B_0$ in period 0.

Even though banks are ex-ante identical, we allow for different initial default decisions *if they are indifferent between default or not at time* 0. Allowing for this heterogeneity will turn out to be important to guarantee existence of a general equilibrium. We denote by $\phi \in [0, 1]$ the fraction of defaulting banks in the initial period. Note that the value of ϕ must be consistent with the optimal decisions of banks, so

$$\phi = \begin{cases} 1 \text{ if } B_0 > \gamma_{-1} p_0 \bar{K}, \\ 0 \text{ if } B_0 < \gamma_{-1} p_0 \bar{K}, \\ \in [0,1] \text{ otherwise.} \end{cases} \tag{13}$$

where γ_{-1} is as discussed in footnote 13.

We let K_t^D and K_t^R denote the capital holdings (per bank) of defaulting and non-defaulting banks in period t. Using that a bank either defaults in the initial period, or it never does, we have the following market clearing condition for capital

$$\phi K_t^D + (1-\phi) K_t^R = \bar{K} \tag{14}$$

for all $t > 0$, with initial condition $K_0^D = K_0^K = \bar{K}$.

Given a sequence of prices $\{p_t\}_{t=0}^{\infty}$ and a sequence of borrowing limits $\{\gamma_t\}_{t=-1}^{\infty}$, let $\mathcal{B}_t$ and $\mathcal{K}_{t+1}^R$ be the policy functions for the repaying banks; and $\mathcal{K}_{t+1}^D$ be the policy function for the defaulting banks. Then, we have the following law of motion for the debt and capital levels:

$$B_{t+1} = \mathcal{B}_{t+1}((\bar{z} + p_t) K_t^R - RB_t) \tag{15a}$$

$$K_{t+1}^R = \mathcal{K}_{t+1}^R((\bar{z} + p_t) K_t^R - RB_t) \tag{15b}$$

$$K_{t+1}^D = \mathcal{K}_{t+1}^D((\underline{z} + p_t) K_t^D) \tag{15c}$$

for all $t \geq 0$. We can now proceed to define a competitive general equilibrium.

Definition 2 (General Equilibrium)**.** A competitive equilibrium given identical initial debt levels, B_0, and capital holdings, $\bar{K}$, is a sequence of prices of capital, $\{p_t\}_{t=0}^{\infty}$, a sequence of borrowing limits, $\{\gamma_t\}_{t=-1}^{\infty}$, a sequence of debt and capital holdings (per bank), $\{B_t, K_t^R, K_t^D\}_{t=0}^{\infty}$, and an initial share of defaulting banks, ϕ, such that

(i) The evolution of debt and capital holdings follow equations (15) where $\mathcal{B}_t$ and $\mathcal{K}_{t+1}^R$ and $\mathcal{K}_{t+1}^D$ represent the policy functions that solves the banks problem in repayment and default respectively given the sequence of prices and borrowing limits;

(ii) The borrowing limits (given the sequence of prices) are equilibrium consistent, that is, Definition 1 is satisfied;

(iii) Markets clear, that is equation (14) holds for all t; and

(iv) The share of defaulting banks, ϕ is consistent with bank's optimality. That is, equation (13) holds.

Before moving on to characterize general equilibrium given any initial condition, we first discuss stationary equilibria, that is, where the capital price and the borrowing limit are constant.

2.4 Stationary Competitive Equilibrium

We define *a stationary equilibrium* as a competitive equilibrium where $p_t = p$, $\gamma_{t-1} = \gamma$, $K^D_{t+1} = K^D$, $K^R_{t+1} = K^R$ and $B_{t+1} = B$ for all $t \geq 0$.

Given a stationary price p, let $R^k(p) \equiv \frac{\bar{z}+p}{p}$ and $R^D(p) \equiv \frac{\underline{z}+p}{p}$ define the returns to capital under repayment and default as before, but this time we make explicit the dependence on the capital price p. Similarly, let $H(\gamma, p)$ be redefined as:

$$H(\gamma, p) \equiv 1 - \left(\frac{R^k(p)/R - \gamma}{R^D(p)/R} \right)^{\frac{1}{\beta}}. \tag{16}$$

The following proposition establishes that there are two types of stationary equilibria.

Proposition 2 (Types of Stationary Equilibria)**.** *Stationary equilibria can be of the following two types:*

(i) **Default equilibrium.** *Let (p^D, γ^D) be a pair such that*

$$\gamma^D = H(\gamma^D, p^D) \tag{17}$$

$$p^D = \frac{\beta}{1-\beta}\underline{z} \tag{18}$$

where γ^D is lowest solution in $[0,1]$ to (17) given p^D.

Such a solution exists (and is unique) if and only if

$$\frac{\bar{z}}{\underline{z}} < \frac{R-1}{\beta^{-1}-1} + R^{-\frac{\beta}{1-\beta}}.$$

If this condition is satisfied and $B_0 \geq \gamma^D p^D \bar{K}$, there exists a stationary equilibrium where $\phi = 1$, $K^D_{t+1} = \bar{K}$, $K^R_{t+1} = 0$, $B_{t+1} = 0$, $p_t = p^D$ and $\gamma_{t-1} = \gamma^D$ for all $t \geq 0$. Banks' dividend payouts are given by $c_t = \underline{z}\bar{K}$.

(ii) **Repayment equilibrium.** *Let (p^R, γ^R) be a pair such that*

$$\gamma^R = H(\gamma^R, p^R) \tag{19}$$

$$p^R = \frac{\beta\bar{z}}{1-\beta-(1-\beta R)\gamma^R} \tag{20}$$

where γ^R is lowest solution in $[0,1]$ to (19) given p^R. Such a solution always exists and is unique.

If $B_0 = \gamma^R p^R \overline{K}$, then there exists a stationary equilibrium in which $\phi = 0$, $K^R_{t+1} = \overline{K}$, $B_{t+1} = B_0$, $K^D_{t+1} = 0$, $p_t = p^R$ and $\gamma_{t-1} = \gamma^R$ for all $t \geq 0$. Banks' dividend payouts are given by $c_t = \bar{z}\overline{K} - (R-1)B_0$.

Proof. In Appendix B.1 □

This proposition says that under one condition on the productivity difference between repayment and default, if the initial level of debt is above some threshold, there is a stationary equilibrium in which all banks default. Likewise, there is a level of debt such that there is a stationary equilibrium in which all banks repay. In this second type of equilibria, the price of capital is higher because it reflects the higher productivity of capital under repayment and the ability to leverage in equilibrium.

The proposition also establishes that for some parameter values, a stationary default equilibrium may fail to exist. This occurs because if all banks were to default, the price of capital would be so low that the return to equity for a bank that did not default would be large enough that there would be no finite borrowing limit and therefore banks would prefer repayment. On the other hand, a repayment stationary equilibrium always exists.

Comparison of stationary equilibria. Let us now compare the two potential stationary outcomes. Note first that $p^R > p^D$, a result that follows immediately from $\beta R \leq 1$, $\gamma^R \geq 0$, and $\bar{z} > \underline{z}$. Intuitively, the demand for capital in the repayment stationary equilibrium is higher than under the default one, as banks have higher productivity and capital serves, in effect, a role as collateral. Notice also that if $\beta R = 1$, we have $R^k = R$ and the borrowing constraint does not bind. In this case, the steady state price reflects only the productivity return and is the same as the one that would prevail in the absence of the limited commitment friction.

By Corollary 1, the result that $p^R > p^D$ implies that $\gamma^D > \gamma^R$. However, we would like to compare the total amount of borrowing that a bank can make per unit of the *value* of its capital, γp. Towards this end, let us define the debt threshold levels implicit in Proposition 2 that characterize the two types of equilibria. Given (γ^D, p^D) and (γ^R, p^R) as defined in Proposition 2, we let

$$\overline{B}^D \equiv p^D \gamma^D \overline{K},$$
$$\overline{B}^R \equiv p^R \gamma^R \overline{K}.$$

That is, $\overline{B}^D$ denotes the debt level at which banks are indifferent between repaying and defaulting when the equilibrium price is constant at p^D. By the same token, $\overline{B}^R$ denotes the debt level at which banks are indifferent between repaying and defaulting when the price of capital is constant at p^R.

We now examine whether the debt level that makes a bank indifferent between repaying and defaulting is higher in the stationary equilibrium with repayment or in the stationary equilibrium with default. We have the following result

Proposition 3. *If the default stationary equilibrium exists, then* $\bar{B}^D > \bar{B}^R$.

Proof. In Appendix B.2. □

In a repayment equilibrium, the debt threshold must be lower than in a default equilibrium. Intuitively, since the return on capital for a repaying bank is lower in the repayment equilibrium, banks must have a lower debt to keep them indifferent between repaying and defaulting.

Ruling out multiplicity. The result that $\bar{B}^D > \bar{B}^R$ is important because if the inequality was reversed, the economy will necessarily feature multiple equilibria (even absent bank runs). In particular, if $\bar{B}^D < B_0 < \bar{B}^R$, the default equilibrium and the repayment equilibrium would both be possible outcomes. That is, if all banks were to repay, asset prices would be high, and an individual bank would choose to repay, while if all banks were to default, asset prices would be low, and an individual bank would choose to default.

We highlight that the fact that default is a dynamic strategic choice is critical to generate a unique equilibrium in our setup. An alternative setup in which default is determined exclusively by the value of the net worth—in particular by whether net worth is positive or negative—would lead to multiplicity as long as the price under repayment is higher than the price under default. This occurs because for a range of debt levels, net worth would be positive under the repayment price but negative under the default price.[18] Instead, in our setup, the default decision depends not only on net worth but also on the sequence of returns.

The absence of multiplicity in this version of the model helps to distinguish our framework from the one in Gertler and Kiyotaki (2015). There, a good equilibrium where banks repay coexist with a bad equilibrium where asset prices fall, net worth turns negative and banks are unable to continue operating.[19] (In terms of our model, this is a scenario where $\bar{B}^R > \bar{B}^D$ and there is a switch of the two thresholds in Figure 2.) Under this situation, the run is fundamental. An individual investor does not have incentives to roll over deposits knowing that the bank will default, regardless of the decision of other investors. The key difference with the runs we will

[18] That is, there exists a debt level b_0 such that $(\bar{z} + p^D)\bar{K} - b_0 < 0 < (\bar{z} + p^R)\bar{K} - b_0$.

[19] One can show that a bank cannot satisfy the incentive compatibility constraint when net worth is negative under the assumption that the portfolio returns do not lead to an infinite value of the bank.

consider in the Section 3 is that the bank runs we consider are self-fulfilling *at the level of the indivual bank*. As will see, this has distinctive implications for policies.

2.5 Transitional Dynamics

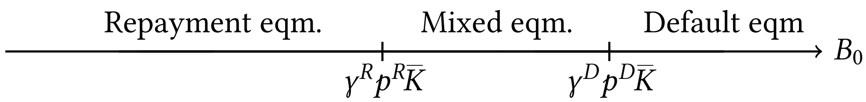

Figure 2: Types of equilibrium depending on B_0

Until now, we have examined stationary equilibrium. The question we address now is how the economy evolves when it does not start at the levels of debt that belong to the two stationary equilibria.

We can distinguish three distinct cases of convergence depending on the initial values of debt relative to $\overline{B}^R$,$\overline{B}^D$.

1. Convergence to repayment equilibrium if $B_0 < \overline{B}^R$. We start by considering the case in which the economy starts with a low level of debt. Specifically, we consider an initial value of debt that is below the stationary values for the repayment and default equilibrium.

Let us consider the case in which $\beta R < 1$. When debt is below $\overline{B}^R$, we conjecture that the dynamics are as follows. For T periods, the return to capital is exactly R, aggregate net worth decreases at rate βR, and the borrowing constraint does not bind. In period T, the borrowing constraint binds, the return to capital is higher than R, and the economy remains at the stationary repayment equilibrium thereafter. Appendix D.1 describes how the value of T and the sequence of prices and debt levels are determined.

Figure 3 illustrates the transition dynamics for B_t and p_t. The note in the figure describes the parameter values used. Panel (a) shows the transition map for B_t. The vertical lines correspond to the different debt threshold levels. The solid blue line shows the corresponding B_{t+1} given a B_t in the horizontal axis. The dashed line shows a particular initial point B_0 and its transition towards the steady state level $\overline{B}^R$. In this case, convergence is achieved in three periods, and debt is increasing along the path. Although not shown, net worth is decreasing too. Panel (b) displays how the price of capital is decreasing in the debt level.

2. Convergence to default equilibrium if $B_0 > \overline{B}^D$. This case is already covered in Proposition 2 and there are, in effect, no transitional dynamics. That is, we have $p_t = p^D$ for all $t \geq 0$, and all banks default in the initial period.

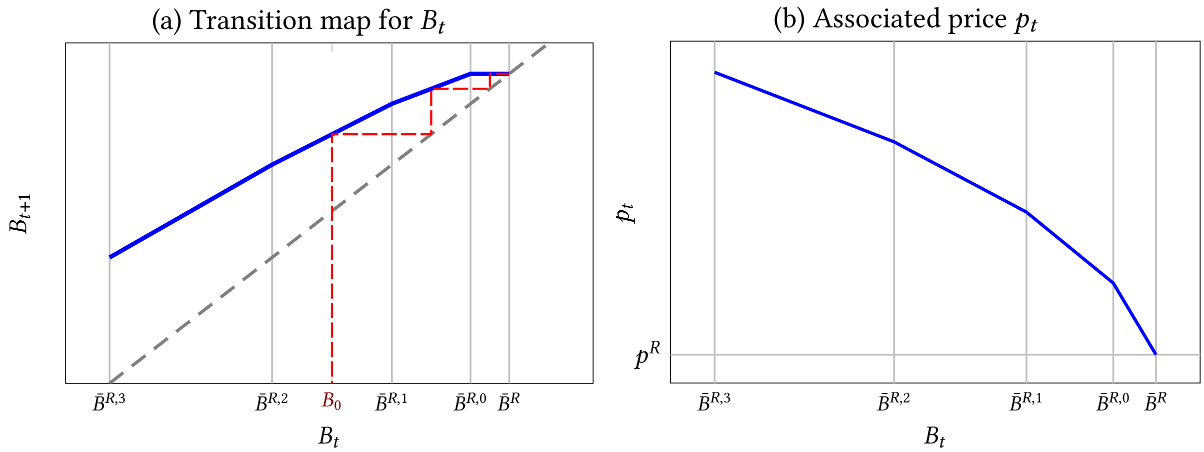

Figure 3: Transition Dynamics in General Equilibrium

Note: This simulation was generated with the following parameters: $R = 1.01$, $\beta = 0.95$, $\bar{z} = 1.5$, $\underline{z} = 1.1$, and $\bar{K} = 1$.

3. Transition if $\bar{B}^D > B_0 > \bar{B}^R$. Consider now the case in which debt is above the stationary level for the repayment equilibrium but below the threshold for the default equilibrium. We argue that in this case, the equilibrium must be non-degenerate. Why does a degenerate equilibrium fail to exist? Under a price consistent with repayment by all banks, an individual bank would find it optimal to default. Conversely, under a price consistent with default by all banks, an individual bank would find it optimal to repay.

We can construct, however, equilibrium where banks are indifferent between defaulting and repaying and such that a fraction ϕ of banks default in the initial period. That is,

$$V_0^D(\bar{K}) = V_0^R((\bar{z} + p_0)\bar{K} - RB_0), \tag{21}$$

where V_0^D and V_0^R are defined respectively in (7) and (10). Recall that V_0^D is a function of the sequence of $\{p_t\}$, and V_0^R is a function of the sequence of $\{p_t\}$ and $\{\gamma_t\}$.

As it turns out, it is possible to characterize this mixed equilibrium in a dynamic system with two variables, given ϕ. The two variables are the fraction of capital owned by banks in repayment, and the debt of banks in repayment as a fraction of the capital shock. Proposition 4 presents the dynamic system, establishes uniqueness and characterizes the resulting allocations (imposing that the borrowing constraint binds along the transition).

Proposition 4 (Characterization of dynamic system for $\bar{B}_0 > B_0 > \bar{B}_R$)**.** *Suppose that in a general equilibrium $R_{t+1}^k > R$ for all $t \geq 0$ and $\phi \in (0,1)$. Let $\tilde{k}_t = \frac{(1-\phi)K_t^R}{\bar{K}}$ and $\tilde{b}_t = \frac{(1-\phi)B_t}{\bar{K}}$. Then, $R\tilde{b}_t > (\bar{z} - \underline{z})\tilde{k}_t$ and $p_t > p^D$ for all $t \geq 0$. The evolution of $(\tilde{k}_t, \tilde{b}_t)$ is uniquely determined starting* *from* $(\tilde{k}_0, \tilde{b}_0)$ *by*

$$\tilde{k}_{t+1} = 1 - \beta \left(\frac{\underline{z} + p_t}{p_t} \right) (1 - k_t),$$
$$\tilde{b}_{t+1} = p_t k_{t+1} - \beta \tilde{n}_t,$$

where $\tilde{n}_t = (\bar{z} + p_t)\tilde{k}_t - R\tilde{b}_t$ *and* p_t *is the unique solution to:*

$$\frac{\left[(\bar{z} + p_t)\tilde{k}_t - Rb_t\right]^{1-\beta} \left[p_t - \beta(\underline{z} + p_t)(1 - \tilde{k}_t)\right]^{\beta}}{\beta^{\beta}(\underline{z} + p_t)\tilde{k}_t} = 1.$$

In addition:

(i) Capital holdings of a repaying bank increase over time. That is, $\tilde{k}_{t+1} > \tilde{k}_t$ *for all* $t \geq 0$*, thus implying that* $K^R_{t+1} > K^D_{t+1}$ *for all* $t \geq 0$*.*

(ii) And $c^D_0 > c^R_0$ *where* c^R_0 *and* c^D_0 *represent the dividend payout at* $t = 0$ *for repaying and defaulting banks respectively.*

Proof. In Appendix B.3 □

Proposition 4 uniquely characterizes the behavior of the economy for a given initial condition in which $\tilde{k}_0 = (1 - \phi)$, $\tilde{b}_0 = (1 - \phi)B_0/\bar{K}$. However, for arbitrary values of ϕ, some of the solutions will eventually become invalid, and thus ϕ needs to be chosen as to be consistent with an equilibrium.[20]

In an equilibrium, $\tilde{k}_t$ is increasing over time. This implies that repaying banks are net buyers of capital while defaulting banks are net sellers. Moreover, given that $k_t \in [0, 1]$, this monotonicity implies that $\tilde{k}_t$ must converge. If k_t were to converge to a value less than 1, the dynamic system above requires that p_t converges to p^D. Now, from the system, we have that $\tilde{b}_{t+1} - \beta R\tilde{b}_t = p_t\tilde{k}_{t+1} - \beta(\bar{z} + p_t)\tilde{k}_t$, which converges to $-\beta(\bar{z} - \underline{z})\tilde{k} < 0$. And thus $\tilde{b}$ must eventually be negative, a contradiction. So it must be the case that in an equilibrium, k_t converges to 1, the level where all the capital is owned by repaying banks. Note that this requires that p_t converges to p^R. The economy must converge to the stationary repayment equilibrium.[21]

Proposition 4 also states an additional result that is useful below: the dividend payout of repaying banks is strictly lower than that of defaulting banks.

[20]This requires that $p_t > p^D$, $R\tilde{b}_t > (\bar{z} - \underline{z})\tilde{k}_t$, and $\tilde{k}_t \in [0, 1]$ for all t.

[21]Note also that the multiplicity and cycles uncovered by Gu, Mattesini, Monnet and Wright (2013) in Kehoe-Levine economies is not a feature of our environment.

It is somewhat surprising that general equilibrium requires partial default for intermediate levels of initial aggregate debt. After all, the equilibrium characterizations in Kehoe and Levine (1993) and Alvarez and Jermann (2000) impose that default is not an equilibrium outcome. We highlight, in addition to the difference in environments we have noted before, that the existence proof in Kehoe and Levine (1993) for debt constrained economies rely on the assumption that all agents are initially endowed with strictly positive assets; an assumption that is violated in our environment. As we will see below, the presence of equilibrium default has stark implications for policy.

Numerical illustrations. In Figure 4, we use the results from Proposition 4 to simulate the model under a mixed equilibrium. We consider an initial value of debt 5 percent above the debt threshold $\bar{B}^R$. Given this initial value of debt, we have $\phi = 0.36$ (i.e., 36% of banks default in equilibrium). Panel (a) shows that the price of capital is low initially, but higher than p^D, and then increases monotonically over time until it reaches p^R, the stationary price under repayment. (The two horizontal dashed lines denote the stationary values of the price). Meanwhile, panel (b) shows that the leverage threshold γ_t is high initially and then decreases over time until it reaches γ^R.

The bottom panels in Figure 4 illustrate the differences between repaying and defaulting banks, represented respectively by the straight and dashed red lines. Panel (c) shows that repaying banks invest more capital than defaulting banks panel, as characterized in part (i) of Proposition 4. Despite having lower initial net worth, as shown in panel (d), repaying banks' ability to lever imply that they invest more. Thanks to their higher portfolio return, their holdings of capital and net worth increase over time and relative those of defaulting banks. In the long-run, defaulting banks' holdings of capital converge to zero. Asymptotically, repaying banks take over the entire capital stock and the economy converges to the repayment stationary equilibrium.

In Figure 5, we present results on the transitional dynamics for a range of initial debt levels using the same parameter values as in the previous figure. There are four panels in the figure: (a) the fraction of banks that default ϕ; (b) the initial price of capital p_0; (c) the initial demand of capital for repaying and defaulting banks; and (d) the initial dividend payout for repaying and defaulting banks. For low values of debt, lower than $\bar{B}^R$, denoted with a vertical dashed line, all banks repay ($\phi = 0$). Recall that if $B_0 = \bar{B}^R$, the price is equal to the stationary price p^R and banks are indifferent between repaying and defaulting. As debt increases beyond $\bar{B}^R$, we reach the region characterized by the mixed equilibrium and ϕ increases until $B_0 = \bar{B}^D$ at which point all banks default and the price becomes equal to p^D, the price in the stationary default equilibrium.

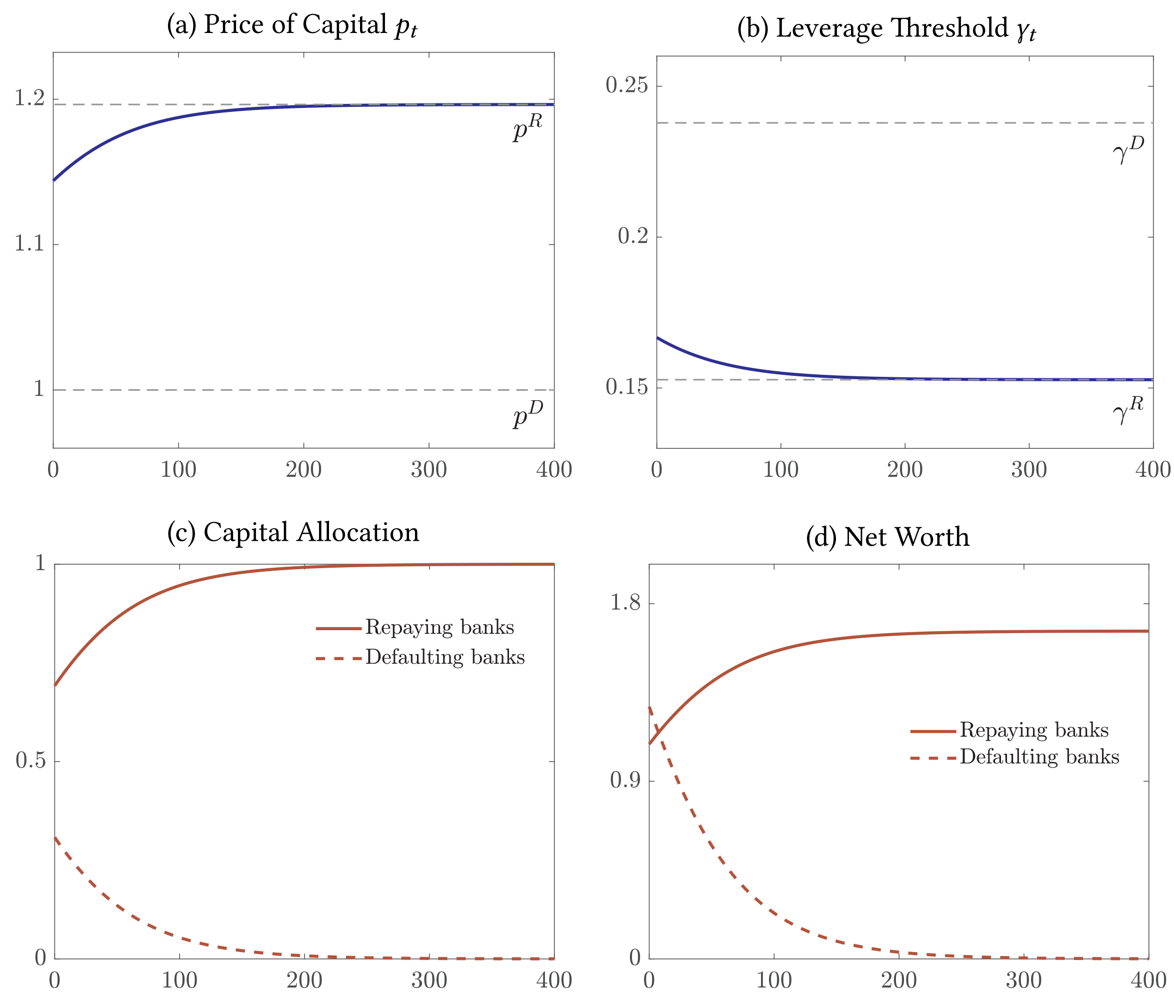

Figure 4: Transition dynamics in a mixed equilibrium

Notes: The simulation was generated using $R = 1.1$, $\beta = 0.97/R$, $\underline{z} = \beta/(1-\beta)$, $\bar{z} = 1.15\underline{z}$, $\overline{K} = 1$ and $B_0 = 0.191$. The x-axis represent periods. The horizontal dashed lines in panels (a) and (b) denote the stationary levels. In panel (c), capital of repaying and default banks is given respectively by $(1-\phi)K_t^R$ and ϕK_t^D. In panel (d), networth of repaying and default banks is given respectively by $(\bar{z}+p_t)K_t^R - RB_t$ and $(\underline{z}+p_t)K_t^R$.

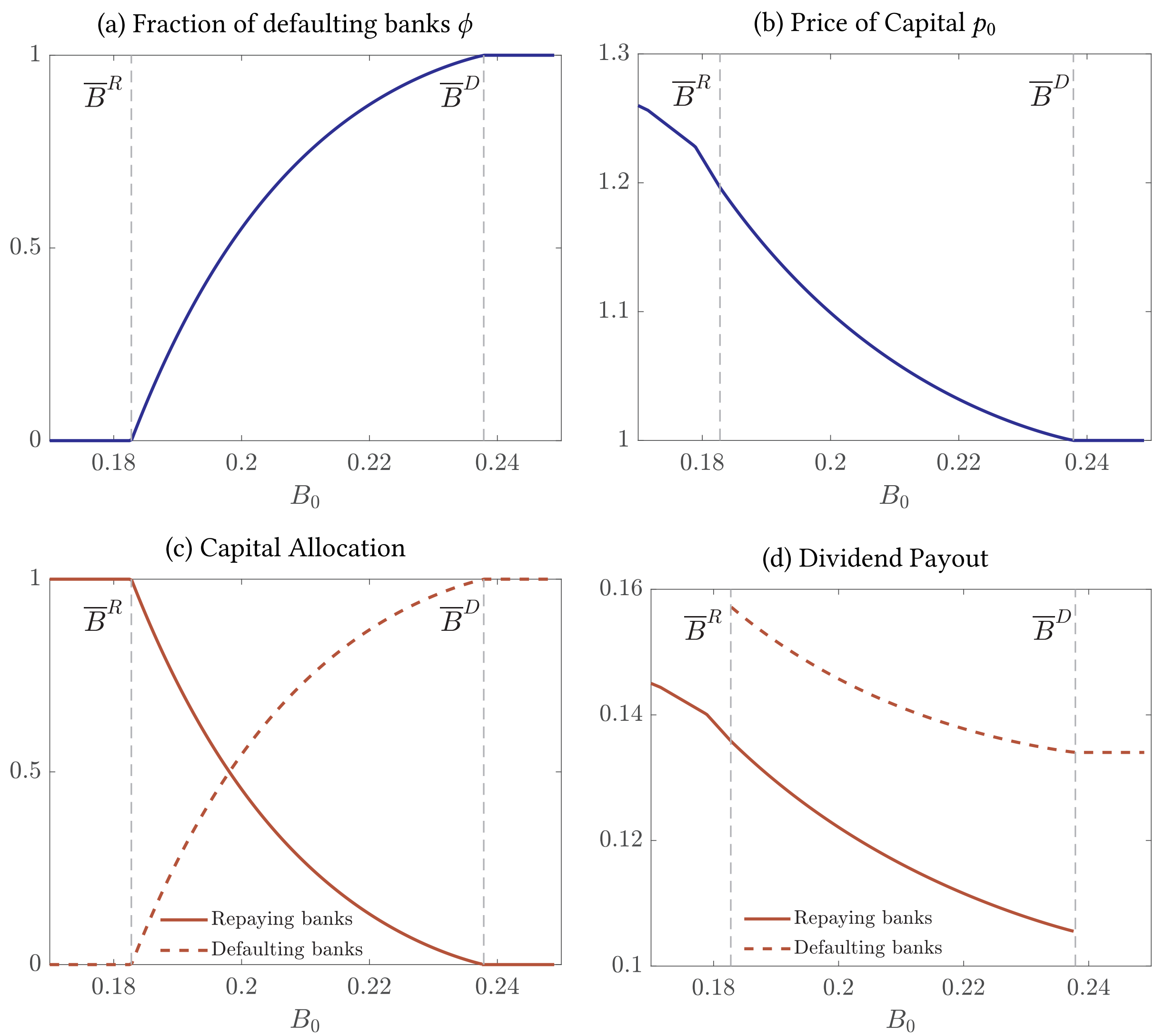

Figure 5: Initial values in transitional dynamics for a range of values of B_0

Notes: The simulation was generated using $R = 1.1$, $\beta = 0.97/R$, $\underline{z} = \beta/(1-\beta)$, $\bar{z} = 1.15\underline{z}$, and $\overline{K} = 1$. The vertical dashed lines denote the stationary borrowing thresholds. In panel (c), capital of repaying and default banks is given by $(1-\phi)K_1^R$ and ϕK_1^D.

3 Bank Runs

In the version of the model we have considered so far, we have abstracted from liquidity considerations. As long as a bank has future cash flows that guarantee repayment, it is able to obtain funding. We now introduce the possibility that banks face a run on deposits and go bankrupt as a result.

We model bank runs as an outcome of a rational expectations equilibrium. We consider a situation in which an individual investor may find it optimal to refuse to roll over deposits when she expects the rest of the investors to do so as well. The details of the game are closest to those in Cole and Kehoe (2000), a workhorse model in the sovereign default literature.[22] We will say that a bank is vulnerable to a run whenever a "panic" by investors that refuse to lend to the bank makes it optimal for the bank to default. We focus on the case under which if a bank is vulnerable to a run, the bank run always takes place.[23]

3.1 Banks' Problem and Borrowing Limits under Bank Runs

As in Section 2, consider a bank that enters the period with good credit standing, k units of capital, and b units of maturing bonds. Given a sequence of prices of capital, the bank's value of default, $V_t^D(k)$, continues to be given by equation (4).

We now introduce the possibility of runs. We use $V_t^{Run}(b,k)$ to denote the value to the bank if it is unable to issue new debt (that is, it suffers a run) and it decides to repay its existing creditors. We will say that a bank is "safe" if even under a run, it chooses to repay its debts rather than default, that is, if $V_t^{Run}(b,k) \geq V_t^D(k)$. We use the term safe because if banks do not find it optimal to default upon a run, investors do not have incentives to run. On the other hand, a bank is "vulnerable" if it finds optimal to default under a run; that is, if $V_t^{Run}(b,k) < V_t^D(k)$.

Thus, given an initial state (b,k), if the bank is safe this period, it cannot suffer a run, and we denote its value by $V_t^{Safe}(b,k)$. If the bank is vulnerable, then we assume that *it suffers a run with probability one,* and thus it defaults (justifying the creditors' beliefs) and attains a value of $V_t^D(k)$.

The value of repayment under a run, $V_t^{Run}(b,k)$, is obtained as follows. Given that the bank cannot issue any new debt, its payments to existing creditors need to come entirely from sales of existing holdings of capital. The bank's dividend payout is therefore given by its net worth minus purchases of new capital. Next period, the bank starts without any debt, and as a result, the

[22] Unlike the Diamond and Dybvig model, Cole and Kehoe does not feature a sequential service constraint. In Cole and Kehoe, investors are atomistic. If all investors refuse to lend and this leads to a default, then an individual investor does not have incentives to lend.

[23] An alternative is to allow for an equilibrium selection involving sunspots as in Cole and Kehoe (2000). In this case, it is possible to have defaults for $t > 0$. However, our assumption allows us to obtain an analytical characterization of the default thresholds.

continuation value is given by the "safe" value function (as a bank with no liabilities cannot suffer a run).[24]

In particular, under a run, the value of repaying for a bank with capital k and debt b can be written as before as just of a function of the net worth. That is, $V_t^{Run}(b,k) = \hat{V}_t^{Run}((\bar{z}+p_t)k - Rb)$ and we have that

$$\hat{V}_t^{Run}(n) = \max_{k' \geq 0, c} \log(c) + \beta V_{t+1}^{Safe}(0, k'), \tag{22}$$

$$\text{subject to}$$

$$c = n - p_t k'.$$

Note that the constraint set in the above problem is non-empty as long as $n \geq 0$ and $V_{t+1}^{Safe}(0,k)$ is defined for any non-negative level of k.

Let us consider the problem of a bank that is safe and decides to repay its debt. Just as in our previous analysis, the bank can issue new bonds as long as its value of repaying tomorrow is higher than or equal to the value of default. Crucially, the next-period value of repayment now needs to be weakly higher than that of default also in the case in which the bank is subject to a run. That is, the bank is subject to the borrowing constraint:

$$\hat{V}_{t+1}^{Run}((\bar{z}+p_{t+1})k' - Rb') \geq V_{t+1}^{D}(k'),$$

The bank chooses a portfolio that guarantees that a run does not occur in the future. Note that if $n < 0$, the bank is necessarily vulnerable to a run.

Thus, when the bank is safe and can obtain funding, it solves a problem analogous to (5), with the difference that to obtain a positive bond price, the bank needs to make sure that it will be safe next period. As in (5), the value of being safe can be written as a function of net worth, $V_t^{Safe}(b,k) = \hat{V}_t^{Safe}((\bar{z}+p_t)k - Rb)$, where $\hat{V}_t^{Safe}$ is given by

$$\hat{V}_t^{Safe}(n) = \max_{n', b', k' \geq 0, c} \log(c) + \beta \hat{V}_{t+1}^{Safe}(n') \tag{23}$$

$$\text{subject to}$$

$$c = n + b' - p_t k'$$

$$n' = (\bar{z}+p_{t+1})k' - Rb' \geq 0$$

$$\hat{V}_{t+1}^{Run}(n') \geq V_{t+1}^{D}(k')$$

[24]A bank with no liabilities can always choose to issue no debt in the future and invest the same amount as a bank that has defaulted at the same level of capital. Because its productivity is strictly higher than a defaulting bank, it follows that $V_t^{Run}(0,k) > V_t^{D}(k)$, and thus a bank without current liabilities is naturally safe.

for $n > 0$, and where we have introduced the constraint $n' \geq 0$, which is a necessary and sufficient condition for a feasible repaying allocation to exist under a run. The last constraint is the borrowing constraint, which as before, also plays the role of the No-Ponzi condition until a further refinement.[25]

If for a given portfolio, $\hat{V}_t^{Safe} < V_t^D$, we say that a bank defaults due to fundamentals. Instead, if $\hat{V}_t^{Run} < V_t^D < \hat{V}_t^{Safe}$, we say that a bank defaults due to runs.[26]

Solution to value functions. The value of default is the same as in Lemma 1. Meanwhile, we can proceed, in a similar fashion to Section 2, to characterize the policy functions and value functions of the bank when it is safe and when it is vulnerable to a run. When the bank is safe and has access to borrowing, we guess that the borrowing constraint in Problem (23) can be written as a linear borrowing constraint $b' \leq \gamma_t p_{t+1} k'$ for some sequence of $\{\gamma_t\}$. Given a sequence of $\{\gamma_t, p_t\}$, the value function $\hat{V}_t^{Safe}$ has the same form as $\hat{V}_t^R$, described in Lemma 4. Note that in equilibrium, however, the sequence $\{\gamma_t\}$ that the bank faces is determined by the condition $\hat{V}_{t+1}^{Run}(n') = V_{t+1}^D(k')$ and thus could be different from the sequence of borrowing limits without runs. Indeed, as we will see below, this implies a tighter borrowing constraint.

We now proceed, accordingly, to characterize the value of repayment under a run.

Lemma 7 (The value of repayment in a run)**.** *Consider a sequence of (strictly positive) prices $\{p_t\}_{t=0}^{\infty}$ and (non-negative) borrowing limits, $\{\gamma_t\}_{t=0}^{\infty}$. that satisfy Condition 2. Then the value of repayment under a run, $\hat{V}_t^{Run}(n)$, and associated policy functions in period t for $n > 0$ are such that:*

(i) Value function:

$$\hat{V}_t^{Run}(n) = A + \frac{1}{1-\beta}\log(n) + \frac{\beta}{1-\beta}\left[\log\left(R_{t+1}^k\right) + \sum_{\tau \geq t+1} \beta^{\tau - t}\log(R_{\tau+1}^e)\right];$$

where A is as in Lemma 1.

(ii) Policy functions:

$$C_t^{Run}(n) = (1-\beta)n, \quad \mathcal{K}_{t+1}^{Run}(n) = \beta\left(\frac{n}{p_t}\right).$$

Proof. In Appendix C.1. □

[25] We do not need to impose the constraint $\hat{V}_{t+1}^{Safe}(n') \geq \hat{V}_{t+1}^D(k')$ in Problem (23). From a simple inspection of the value functions, it is clear that $\hat{V}_{t+1}^{Safe}(n') \geq \hat{V}_{t+1}^{Run}(n')$ and hence the constraint is satisfied if $\hat{V}_{t+1}^{Run}(n') \geq \hat{V}_{t+1}^D(k')$.

[26] This result contrasts with Gertler and Kiyotaki (2015) and Gertler et al. (2020) where a bank defaults when net worth is negative irrespective of whether creditors of the individual bank are willing to roll over or not.

The value function is again log-linear in net worth. The difference relative to $\hat{V}_t^{Safe}$ is that the inability to obtain new deposits lowers the return on net worth in the first period from R^e_{t+1} to R^k_{t+1}, thereby reducing the value from repaying.[27] As long as $\gamma_t > 0$ and $R^k_{t+1} > R$, then $\hat{V}_t^{Safe}(n) > \hat{V}_t^{Run}(n)$. In addition, the bank continues to consume a fraction $1-\beta$ of net worth and invest the rest in capital.

We have the following proposition characterizing the default condition when the bank is subject to a run.

Proposition 5 (Default decision under runs)**.** *Consider a sequence of (strictly positive) prices, $\{p_t\}_{t=0}^{\infty}$, and a sequence of (non-negative) borrowing limits, $\{\gamma_t\}_{t=0}^{\infty}$ that satisfy Condition 2. Then, the value of γ_t that makes a bank indifferent between repayment and default at $t+1$ is such that*

$$\beta \log\left(\frac{\bar{z}+p_{t+2}(1-\gamma_{t+1}R)}{\bar{z}+p_{t+2}}\right) - \beta^2 \log\left(\frac{\bar{z}+p_{t+3}(1-\gamma_{t+2}R)}{\bar{z}+p_{t+3}}\right) + \\ + \beta^2 \log\left(1-\gamma_{t+2}\frac{p_{t+3}}{p_{t+2}}\right) = \log\left(\frac{\bar{z}+p_{t+1}(1-\gamma_t R)}{\underline{z}+p_{t+1}}\right) \quad \text{(G-run)}$$

for all $t \geq 0$.

Proof. In Appendix C.2 □

Using the above, we now can define the equilibrium-consistent borrowing limits with runs given a sequence of prices:

Definition 3. Given a sequence of (strictly positive) prices $\{p_t\}_{t=0}^{\infty}$, we say a sequence of (non-negative) borrowing limits $\{\gamma_t\}_{t=0}^{\infty}$ is *equilibrium-consistent with runs* if Conditions 2, and 3 hold and equation (G-run) is satisfied for all $t \geq 0$.

3.2 General Equilibrium with Runs

The definition of general equilibrium follows exactly the definition in Section 2, except that the borrowing limits must be equilibrium-consistent with runs. That is, given initial debt levels and capital holdings, an equilibrium is a sequence of prices of capital $\{p_t\}_{t=0}^{\infty}$, a sequence of borrowing limits, $\{\gamma_t\}_{t=-1}^{\infty}$, a sequence of (per-bank) aggregate debt and capital levels, $\{B_t, K_t^R, K_t^D\}_{t=0}^{\infty}$, and an initial share of defaulting banks, ϕ, such that (i) the evolution of aggregate debt and capital are consistent with banks' policies (ii) banks optimize, (iii) the market for capital clears, and (iv) borrowing limits are equilibrium-consistent with runs (i.e., eq. (G-run) holds).

[27] If one imposes artificially that $\gamma_t = 0$ in the value function $\hat{V}_t^R$, while making all other subsequent γ's the same, we reach the same value as in $\hat{V}_t^{Run}$.

Stationary equilibria with runs. We define stationary competitive equilibria as before: a situation in which p_t, γ_t, capital allocations and debt are constant for all $t \geq 0$.

We characterize stationary equilibria with runs in a manner similar to that in Proposition 2. Using equation (G-run), we first define a condition that the stationary value of γ must satisfy. That is, $\gamma = H^r(\gamma, p)$ where

$$H^r(\gamma, p) \equiv 1 - \left(1 - \frac{R}{R^k(p)}\gamma\right)^{1+\frac{1-\beta}{\beta^2}} \left(\frac{R^k(p)}{R^D(p)}\right)^{\frac{1}{\beta^2}}$$

Notice that we have emphasized the dependence of the returns on the price of capital by writing $R^k(p)$ and $R^D(p)$.

The function H^r has similar properties to H, defined in (16). In particular, H^r is increasing and strictly concave in γ in $[0, 1)$, $H^r(1, p) \leq 1$, and $H^r(0, p) < 0$ given that $R^k > R^D$. And thus, H^r features at most two fixed points in $[0, 1]$. We have the following result, a version of Lemma 6 for the case with runs:[28]

Lemma 8 (Stationary borrowing limits under a constant price). *Consider a constant price of capital $p > 0$ such that $R^k \geq R$.*

(i) If $\beta R^k/R < \beta + (1-\beta)(\beta R^D/R)^{\frac{1}{1-\beta}}(\beta R^k/R)^{-\beta}$ and $\beta R^D/R < 1$, then there is a unique stationary (equilibrium-consistent under a run) borrowing limit $\gamma^\star$ where $\gamma^\star$ is the smallest solution to $\gamma = H^r(\gamma, p)$ for $\gamma \in [0, 1)$.

(ii) Otherwise, there exists no stationary (equilibrium-consistent under a run) borrowing limit.

Proof. In Appendix C.3 □

Note that the condition for existence in part (i), although quite similar to the condition in Lemma 6 is in effect a weaker one. That is, the economy with runs admits a higher return on capital owing to the fact that the borrowing constraint is tighter.

With this existence result at hand, we can then proceed to characterize the stationary equilibria.

Proposition 6 (Types of stationary equilibria with runs). *Stationary equilibria with runs can be of the following two types:*

[28] Differently from Lemma 6, in this case we cannot show that all equilibrium consistent borrowing limits are stationary. Part of the difficulty arises from characterizing the dynamics of the system described by (G-run), a second-order difference equation that makes the analysis significantly more complex. However, the results in Lemma 8 suffice for characterizing the general equilibrium, as we will see below.

(i) **Default equilibrium.** *Let (p^{rD}, γ^{rD}) be given by a solution to*

$$\gamma^{rD} = H^r(\gamma^{rD}, p^{rD}) \tag{24}$$

$$p^{rD} = \frac{\beta}{1-\beta}\underline{z} \tag{25}$$

where γ^D is lowest solution in $[0,1]$ to (24), given p^{rD}. Such a solution exists (and is unique) if and only if

$$\frac{\bar{z}}{\underline{z}} < \frac{R-1}{\beta^{-1}-1} + \frac{R^{-\frac{\beta}{1-\beta}}}{x_0^{\beta}}$$

where x_0 is the unique solution in $(\beta, 1)$ to $x_0^{\beta}(x_0 - \beta) = (1-\beta)R^{-\frac{1}{1-\beta}}$.

If $B_0 \geq \gamma^{rD} p^{rD} \overline{K}$, there exists a stationary equilibrium in which $\phi = 1$, $K^D_{t+1} = \overline{K}$, $K^D_{t+1} = 0$, $B_{t+1} = 0$, $p_t = p^{rD}$, and $\gamma_{t-1} = \gamma^{rD}$ for all $t \geq 0$. Banks' dividend payouts are given by $c_t = \underline{z}\overline{K}$.

(ii) **Repayment equilibrium.** *Let (p^{rR}, γ^{rR}) be given by the solution to*

$$\gamma^{rR} = H^r(\gamma^{rR}, p^{rR}) \tag{26}$$

$$p^{rR} = \frac{\beta\bar{z}}{1-\beta-(1-\beta R)\gamma^{rR}} \tag{27}$$

where γ^D is lowest solution in $[0,1]$ to (26) given p^{rR}. Such a solution always exists and is unique.

If $B_0 = \gamma^{rR} p^{rR} \overline{K}$, then there exists a stationary equilibrium in which $\phi = 0$, $K^R_{t+1} = \overline{K}$, $B_{t+1} = B_0$, $K^D_{t+1} = 0$, $p_t = p^{rD}$, and $\gamma_{t-1} = \gamma^{rR}$ for all $t \geq 0$. Banks' dividend payouts are given by $c_t = \bar{z}\overline{K} - (R-1)B_0$.

Proof. In Appendix C.4 □

It is useful to define again the threshold debt levels implicit in Proposition 6. That is, given (γ^{rD}, p^{rD}) and (γ^{rR}, p^{rR}), we let

$$\overline{B}^{rD} \equiv p^{rD}\gamma^{rD}\overline{K},$$
$$\overline{B}^{rR} \equiv p^{rR}\gamma^{rR}\overline{K}$$

Let us analyze how the debt thresholds and prices differ between the case without runs and the case with runs. First note that the price in the stationary default equilibrium is the same with and without runs, $p^{rD} = p^D = \beta\underline{z}/(1-\beta)$. Using this result, we can show that the debt threshold determining the default stationary equilibrium is lower with runs.

To examine $\bar{B}^{rR}$, it is useful to distinguish two cases. If $\beta R = 1$, just as in the economy without runs, the borrowing constraint is not binding in the stationary repayment equilibrium, and the price is such that the return to capital and the interest rate are equalized: $R^k = R$. In this case, interestingly, we have that $\gamma^{rR} = \gamma^R$ and thus $\bar{B}^{rR} = \bar{B}^R$. Hence the presence of runs does not affect the threshold for the repayment stationary equilibrium when $\beta R = 1$.[29] If $\beta R < 1$, this result no longer holds. In fact, we can show that γ is strictly lower under runs and therefore the stationary price and the debt threshold is lower under runs. We summarize these results in the following lemma:

Lemma 9. *The following holds:*

(i) If a default equilibrium without runs exists, then a default equilibrium with runs exists, and $\gamma^{rD} < \gamma^D$ *and* $\bar{B}^{rD} < \bar{B}^D$.

(ii) If $\beta R = 1$, *then* $\gamma^{rR} = \gamma^R$, $p^{rR} = p^R$ *and* $\bar{B}^{rR} = \bar{B}^R$. *If* $\beta R < 1$, *then* $\gamma^{rR} < \gamma^R$, $p^{rR} < p^R$ *and* $\bar{B}^{rR} < \bar{B}^R$.

Proof. In Appendix C.5. □

Intuitively, the presence of runs makes the borrowing constraints tighter and this expands the conditions for existence of a stationary default equilibrium expands. The presence of runs leads to borrowing limits that are tighter than the "not too tight" limits of Alvarez and Jermann (2000) that emerged in the case without runs.

Having characterized the potential stationary outcomes in the economy with runs, we now discuss briefly the transitional dynamics.

Transitional dynamics with runs. Just like the case without runs, we can distinguish three distinct regions of convergence depending on the initial values of debt relative to $\bar{B}^{rR}$,$\bar{B}^{rD}$.

1. Convergence to repayment equilibrium with runs if $B_0 < \bar{B}^{rR}$. This case is analogous to the economy without runs and is discussed in Appendix D.2.

[29] In this case, given that $R^k = R$, R^e is independent of the value of γ and also equals R, and thus $\hat{V}^R(n) = \hat{V}^{Run}(n)$. A bank suffering a run cannot leverage and needs to repay its debt. But given that $R^k = R$, this is no different from a bank that does not suffer a run and decides to repay. To the extent that net worth is positive, such a bank could also optimally have chosen to reduce its debt to zero and scale down its capital, as it is indifferent between capital and bonds. This is quite different from the sovereign debt results in Cole and Kehoe (2000), where the possibility of a run does affect the default threshold when $\beta R = 1$. The key is that in our model, when $\beta R = 1$, in the stationary repayment equilibrium, the value of capital represents the present value of the future "endowments" of the bank. Access to a spot liquid market for capital renders the presence of runs irrelevant.

2. Convergence with runs if $B_0 > \bar{B}^{rD}$**.** All banks default immediately, $\phi = 1$, $K_t^D = \bar{K}$, and $p_t = p^{rD}$ for all $t \geq 0$.

3. Transition with runs if $\bar{B}^{rR} > B_0 > \bar{B}^{rD}$**.** As in the economy without runs, we have that the equilibrium must be non-degenerate.[30] An interesting observation here regards the comparison of the policies for a bank facing a run compared to a defaulting bank. Let c_0^{Run} and K_1^{Run} denote the initial dividend payout and capital choices for a bank that is facing a run and decides to repay. We have the following result.

Lemma 10. *Consider an equilibrium with runs where* ϕ *is interior. Then,* $c_0^{Run} < c_0^D$ *and* $K_1^{Run} < K_1^D$*.*

Proof. In Appendix C.6 □

An implication of the lemma is that a repaying bank facing a run is a net seller of capital, and in particular it sells more capital than a defaulting bank. As we will see in the next section, through effects on the price of capital, government policies can have important implications for the vulnerability of banks to self-fulfilling runs.

4 Are defaults inefficient?

We consider a policy in which the government can directly control the default decision of banks in period $t = 0$, but it does not intervene in the economy in any other way. One goal of this exercise is to analyze the extent to which private repayment/default decisions are socially optimal. From a practical standpoint, the analysis will shed light on whether policies like subsidies or debt-forgiveness aimed at preventing defaults are desirable.

Consider starting from an equilibrium in which the share of defaulting banks is interior and denote by ϕ^E the equilibrium share of defaulting banks and by $\{p_0^E, p_1^E, p_2^E, \dots\}$ the sequence of prices.

Suppose now that the government directly controls the share ϕ of defaulting banks at $t = 0$ while banks retain their ability to choose dividends, issue new bonds (as long as the governments commands them to repay) and buy/sell capital. In subsequent periods, we assume that the default decision (and all future choices) are done by the banks. That is, banks in subsequent periods default if and only if the value function of default is lower than the value of repayment. This implies that the equilibrium consistency of borrowing limits remain as in our baseline economy.

[30] In this case, we solve the model numerically by searching for the sequence of $\{\gamma_t, p_t\}$ and ϕ that satisfy market clearing condition (14), the initial indifference condition for repaying/defaulting $V_0^{Run} = V_0^D$ and the dynamic equation for γ, (G-run).

The problem of a repaying bank at time $t = 0$ starting with initial debt $b_0 = B_0$ and initial capital holdings $k_0 = \bar{K}$ is

$$V^R = \max_{k' \geq 0, b', c} \{u(c) + \beta V_1^R(k', b'; \{p_1, p_2, \dots\})\}, \tag{28}$$

$$\text{subject to}$$
$$c \leq (\bar{z} + p_0)\bar{K} - RB_0 + b' - p_0 k',$$
$$b' \leq \gamma_0(\{p_1, p_2, \dots\}) p_1 k'.$$

The value for a defaulting bank is

$$V^D = \max_{k' \geq 0, c} \{u(c) + \beta V^D(k'; \{p_1, p_2, \dots\})\}, \tag{29}$$

$$\text{subject to}$$
$$c \leq (\underline{z} + p_0)\bar{K} - p_0 k'.$$

Using these value functions, we denote total bank welfare by

$$W = (1 - \phi)V^R + \phi V^D \tag{30}$$

Our analysis will focus on the welfare for banks. As is well-known, the fact that default generates deadweight losses can make an ex-post renegotiation between a borrower and creditors mutually desirable. The reason why we focus on banks' welfare is to isolate a novel pecuniary externality that emerges in our framework. One potential interpretation for leaving aside creditors' welfare is that creditors are foreign and the planner therefore puts no weight on their welfare.[31]

4.1 A partial analysis.

The government policy for ϕ affects the demand for capital as defaulting and repaying banks have different demand for capital. Thus, the policy potentially affects the market clearing capital prices of capital in period $t = 0$ as well as subsequent periods.

To be able to obtain some analytical insights, let us consider a partial scenario in which the changes in ϕ do not affect the prices in periods $t \geq 1$. That is, we take those future prices as given, but maintain that p_0 clears the capital market in $t = 0$. Notice that an implication of this assumption is that we also take as given $\{\gamma_t\}_{t \geq 0}$. Let $V^R(p_0)$ ad $V^D(p_0)$ denote the associated repayment and default value functions, and $k^R(p_0)$ and $k^D(p_0)$ represent the demand for capital

[31]The welfare of creditors is given by $(1 - \phi)B_0$, given the assumption that they have linear utility. If the planner were to put positive weight on creditor's utility, it would trade off the losses from creditors with the gains by banks, which will become clear below.

of repaying and defaulting banks. Market clearing implies that

$$(1-\phi)k^R(p_0) + \phi k^D(p_0) = \overline{K}.$$

A key element that we turn next is how the initial asset price changes in response to the government policy for ϕ.

Let us consider a case where $R_1^k > R$, so that the borrowing constraint is binding in the first period. Assuming differentiability of the policy functions with respect to p_0 (which we show below), we have that

$$\frac{dp_0}{d\phi} = \frac{k^R(p_0) - k^D(p_0)}{(1-\phi)\frac{dk^R(p_0)}{dp_0} + \phi\frac{dk^D(p_0)}{dp_0}}. \tag{31}$$

At the starting competitive equilibrium allocation with p_0^E, we have that $k^R(p_0^E) > k^D(p_0^E)$ by Proposition 4. That is, repaying banks demand more capital than defaulting ones (and the numerator in (31) is positive).

The denominator in (31) corresponds to the change in the demand for capital in response to a change in ϕ. To see that the demand for capital is decreasing in p_0 notice that we have

$$\frac{\partial k^D(p_0)}{\partial p_0} = -\beta\frac{\overline{K}z}{p_0^2} < 0,$$

and that

$$\frac{\partial k^R(p_0)}{\partial p_0} = -\frac{\beta\left[(\bar{z}+\gamma_0 p_1)\overline{K} - RB_0\right]}{(p_0-\gamma_0 p_1)^2}, \tag{32}$$

which is negative evaluated at $p_0 = p_0^E$ using the fact from Proposition 4 that $k^R(p_0^E) > \overline{K}$. [32]

With these two results on hand, we have that

$$\left.\frac{dp_0}{d\phi}\right|_{\phi=\phi^E} < 0$$

That is, a larger share of defaulting banks leads to a decrease in the price of capital. Intuitively, by increasing the share of defaulting banks, the government shifts the composition towards banks with lower demand for capital. To the extent that the demand for capital is downward sloping,

[32] To see formally that the numerator in (32) is positive, note that using the budget constraint we have

$$0 < c_0^R = (\bar{z}+p_0)\overline{K} - RB_0 - (p_0-\gamma_0 p_1)k^R(p_0) < (\bar{z}+p_0)\overline{K} - RB_0 - (p_0-\gamma_0 p_1)\overline{K} = (\bar{z}+\gamma_0 p_1)\overline{K} - RB_0,$$

where the second inequality follows from $p_0 > \gamma_0 p_1$, based on Lemma 3 and $p_0 > p_D$ (the latter implying that $k^R(p_0) > \overline{K}$). Effectively, repaying banks are net buyers of capital. Both income and substitution effects lead to a reduction in their demand for capital when its price increases.

market clearing requires an equilibrium reduction in p_0.

Let us now turn to banks' welfare. Computing the derivative (30) with respect to ϕ at the competitive allocation, we obtain:

$$\left.\frac{dW}{d\phi}\right|_{\phi=\phi^E} = (V^D(p_0^E) - V^R(p_0^E)) + \left[\phi \left.\frac{dV^D(p_0)}{dp_0}\right|_{p_0=p_0^E} + (1-\phi)\left.\frac{dV^R(p_0)}{dp_0}\right|_{p_0=p_0^E}\right]\frac{dp_0}{d\phi}.$$

The second term in this expression involves the derivatives of the value functions with respect to the initial asset price. Using the envelope condition on the repaying and defaulting bank problems, we obtain

$$\left.\frac{dV^R(p_0)}{dp_0}\right|_{\phi=\phi^E} = u'(c^R)(\bar{K} - k^R(p_0^E)), \text{ and } \left.\frac{dV^D(p_0)}{dp_0}\right|_{\phi=\phi^E} = u'(c^D)(\bar{K} - k^D(p_0^E)).$$

where c^R and c^D denote the dividend payout of banks that repay and default at the equilibrium allocation. Using these conditions and imposing the market clearing condition at $t = 0$, we obtain that

$$\left.\frac{dW}{d\phi}\right|_{\phi=\phi^E} = (V^D(p_0^E) - V^R(p_0^E)) - (1-\phi^E)\left[u'(c^R) - u'(c^D)\right](k^R(p_0^E) - \bar{K})\left.\frac{dp_0}{d\phi}\right|_{\phi=\phi^E}. \tag{33}$$

This expression characterizes how banks' welfare changes in response to a government policy of varying the share of defaulting banks (while keeping future prices constant). We now distinguish between an economy without runs and with runs.

The case without runs. Starting from an equilibrium in which ϕ^E is interior, the first term in (33) is zero. That is, in the absence of runs, we have that banks are indifferent between repaying and defaulting and $V^D = V^R$. Regarding the second term in (33), we have that $u'(c^R) - u'(c^D) > 0$ by Proposition 4. In addition, based on the arguments above, we have that $(k^R(p_0) - \bar{K})\frac{dp_0}{d\phi}\Big|_{\phi=\phi^E} < 0$. Thus, starting from the competitive equilibrium with $\phi^E \in (0, 1)$, *the planner will find it optimal to increase the share of defaulting banks.*

The intuition for this result is as follows. When the planner increases ϕ, there are two effects to consider, per equation (33). The first is related to the difference in the value functions between repaying and defaulting banks. In principle, this could generate a loss as increases in ϕ force a repaying bank to choose a sub-optimal decision. However, in the equilibrium without runs, this effect is exactly zero at the margin, as banks are indifferent between repaying and defaulting.

But there is an additional channel that arises through the impact on the equilibrium price p_0. When the planner increases ϕ, the demand for capital falls, as defaulting banks have a lower capital demand than repaying ones. This requires that the price of capital falls to clear the market. This reduction in the price of capital redistributes from net sellers to net buyers—that is, from defaulting to repaying banks. Because in equilibrium, defaulting banks have a higher dividend payout level in the first period, this redistribution is beneficial and increases banks' welfare.

The case with runs. The key difference in the presence of runs is that the first term in (33) is no longer zero. The defaulting bank has a value that is strictly lower than that of a repaying bank. In this case, V^{Run} does not correspond to the value function of a repaying bank in equilibrium. Rather, the value of a repaying bank in equilibrium is V^{Safe}.[33] We thus have that $V^R = V^{Safe} > V^{Run} = V^D$ and the first term is negative.

The fact that the first term in equation (33) is strictly negative implies that there is a first-order loss that arises from forcing a safe bank to default. In this case, banks are defaulting because of the run, but would be otherwise better off repaying if investors were willing to roll over the deposits. Thus, if the planner can reduce the share of defaulting banks, this would shift the composition of banks towards higher values and generate a first-order welfare gain. Thus, it is possible that the inefficiency generated by the coordination failure between investors is enough to guarantee that the planner would like to *reduce* the share of defaulting banks rather than to increase it, as before. Notice that a lower share of defaulting banks increases also the welfare of creditors and so the policy can be Pareto improving.[34]

4.2 Numerical Results

In the above exercises, we have kept the capital prices from period $t = 1$ onward constant. In this way, we were able to obtain analytical results highlighting how the planner's policy of changing the share of defaulted banks affected the capital price in the first period and banks' welfare. In general, however, this policy will also affect the capital prices in subsequent periods. To be able to see what happens in this case, we turn to numerical simulations.

In Figure 6, we contrast the results of the government policy for the economy with fundamentals-driven default and for the one with run-driven default. We consider a share of defaulting banks ranging from 0% to 100% and illustrate the competitive outcome with a solid dot.

The figure shows that in the economy without runs, the maximum welfare is achieved with a higher share of defaulting banks relative to the competitive outcome. One can also see in panel (b) that the policy results in a lower equilibrium price, facilitating a transfer from the low

[33] Recall that $V_t^{Run} < V_t^{Safe}$ as long as $R_{t+1}^k > R$ and $\gamma_t > 0$.

[34] On the other hand, absent runs, a policy of increasing the share of defaulting banks improves banks' welfare but lowers the welfare of creditors.

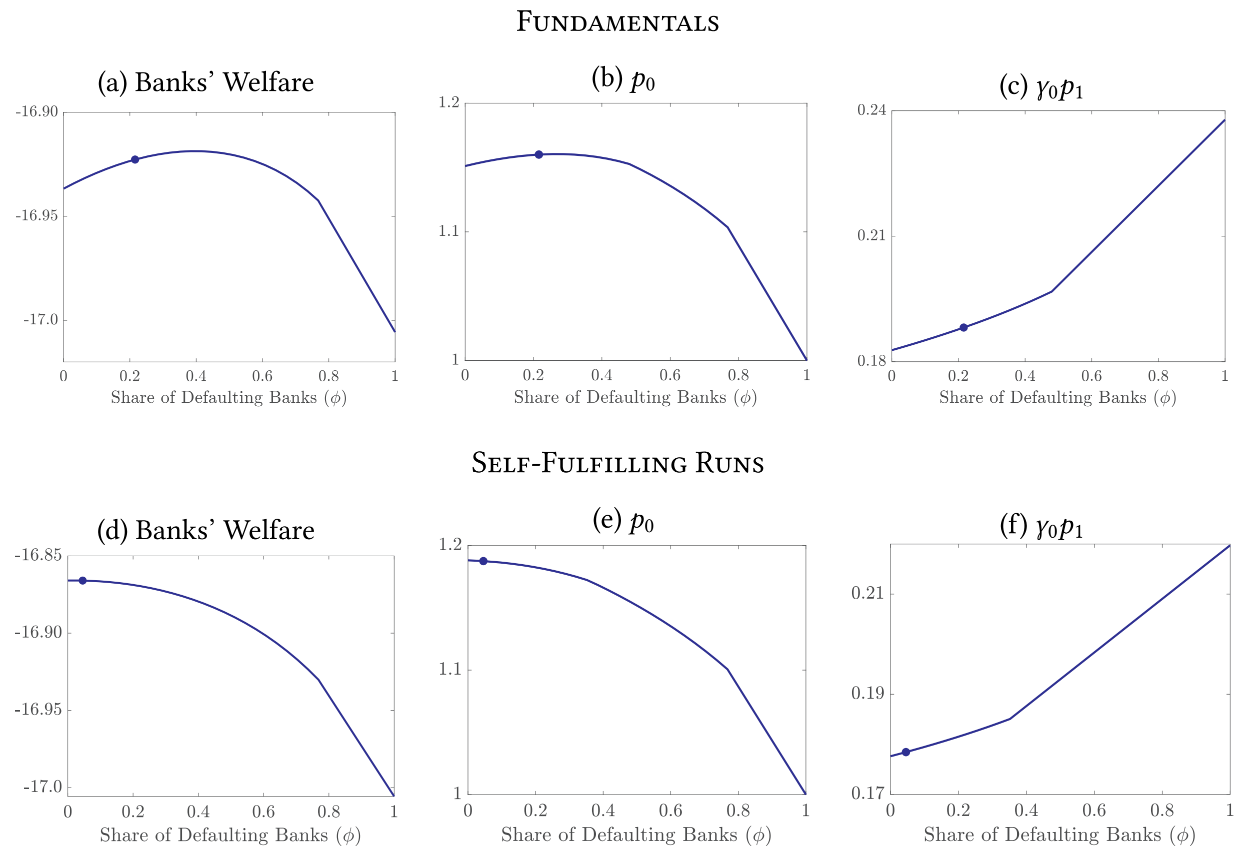

Figure 6: Policy of Choosing Share of Defaulting Banks

Notes: The simulation was generated using $R = 1.1$, $\beta = 0.97/R$, $\underline{z} = \beta/(1-\beta)$, $\bar{z} = 1.15\underline{z}$, and $\bar{K} = 1$. The values for initial debt are given by $B_0 = \alpha\bar{B}^{rR} + (1-\alpha)\bar{B}^{rD}$ and $B_0 = \alpha\bar{B}^{R} + (1-\alpha)\bar{B}^{D}$ with $\alpha = 0.97$, respectively for the economies with and without runs. The solid dot denotes the competitive outcome.

marginal utility defaulting banks to the high marginal utility repaying banks. Panel (c) shows a result that is not highlighted in the analytical result in equation (33). A larger share of defaults increases leveraged returns and raises the amount that banks can borrow. The latter is an effect not internalized by banks, which leads the government to choose an even larger share of defaults.

On the other hand, under run-driven defaults, the government finds it optimal to reduce the share of defaulting banks. In this example, the optimal amount of defaults is zero, as illustrated by panel (d).

Discussion on lender of last resort The fact that that defaults can be excessive due to self-fulfilling runs suggest the importance of lender of last resort policies. A novel implication of our general equilibrium analysis is that for lender of last resort policies to be effective, they must cover a significant share of the financial system. To fix ideas, consider an equilibrium where 20% of the banks will face a run and default. A government guarantee to provide liquidity to this specific subset of institutions would be successful at protecting them from runs. However, since ϕ is determined in general equilibrium, other banks would now be facing runs. Thus, despite the government policy being successful at protecting this subset, there would still be 20% of banks defaulting. This result may indeed shed some light on why during the 2008 financial crisis, the financial system was vulnerable to runs despite many banks having access to liquidity support from the Federal Reserve.

5 Conclusions

We developed a framework to investigate the (in)efficiency of banks' default decisions. Our findings challenge the widespread view that policies should be directed towards averting defaults. We show that while defaults may be excessive under self-fulfilling runs, the competitive equilibrium features too little defaults when defaults are triggered by fundamentals.

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Online Appendix to "Bank Runs, Fragility, and Credit Easing"

By Manuel Amador and Javier Bianchi

A Proofs for Section 2.1–2.2 (Partial Equilibrium)

A.1 Proof of Lemma 1

Proof. The problem of a bank under default facing a sequence of prices $\{p_t\}_{t=0}^{\infty}$ is given by

$$V_t^D(k) = \max_{k',c} \log(c) + \beta V_{t+1}^D(k') \quad \text{(A.1)}$$

$$\text{subject to}$$

$$c = (p_t + \underline{z})k - p_t k'$$

We conjecture that

$$V_t^D(k) = \mathbb{B}_t^D + \frac{1}{1-\beta}\log(k(\underline{z}+p_t)) \quad \text{(A.2)}$$

Replacing this conjecture into (A.1) and substituting out consumption from the budget constraint, we have that

$$V_t^D(k) = \max_{k'} \log(\underline{z}k + p_t(k-k')) + \beta\left[\frac{1}{1-\beta}\log(k'(p_{t+1}+\underline{z})) + \mathbb{B}_{t+1}^D\right] \quad \text{(A.3)}$$

The first-order condition with respect to k' is given by

$$\frac{p_t}{\underline{z}k + p_t(k-k')} = \left(\frac{\beta}{1-\beta}\right)\frac{1}{k'}$$

$$\Rightarrow \; k' = \frac{\beta(\underline{z}+p_t)}{p_t}k \quad \text{(A.4)}$$

By the method of undetermined coefficients, we can now verify the conjecture and solve for $\mathbb{B}_t^D$. We substitute (A.4) into the right-hand side of (A.3) and replace the conjectured guess for $V_t^D(k)$ on the left-hand side of (A.3).

$$\mathbb{B}_t^D + \frac{1}{1-\beta}\log((\underline{z}+p_t)k) = \log\left((1-\beta)(\underline{z}+p_t)k\right) + \beta\left[\frac{1}{1-\beta}\log\left(\beta R_{t+1}^D(\underline{z}+p_t)k\right) + \mathbb{B}_{t+1}^D\right]$$

where we have used the definition of R_{t+1}^D. Rearranging this equation, we can observe that the terms multiplying $\log(k)$ cancel out. After simplifying, we obtain that the conjectured value function is verified when $\mathbb{B}_t^D$ satisfies:

$$\mathbb{B}_t^D = \log(1-\beta) + \frac{\beta}{1-\beta}\log(\beta) + \frac{\beta}{1-\beta}\log\left(R_{t+1}^D\right) + \beta\mathbb{B}_{t+1}^D \quad \text{(A.5)}$$

Iterating forward on this equation and imposing $\lim_{\tau\to\infty}\beta^\tau \log\left(R^D_{\tau+1}\right)=0$, as in Condition 1, we have

$$\mathbb{B}^D_t = \frac{1}{1-\beta}\left[\frac{\beta}{1-\beta}\log(\beta)+\log(1-\beta)\right]+\frac{\beta}{1-\beta}\sum_{\tau\geq t}\beta^{\tau-t}\log\left(R^D_{\tau+1}\right) \tag{A.6}$$

Replacing (A.6) in (A.2), we obtain that the value under default is given by

$$V^D_t(k) = A + \frac{1}{1-\beta}\log((\underline{z}+p_t)k)+\frac{\beta}{1-\beta}\sum_{\tau\geq t}\beta^{\tau-t}\log\left(R^D_{\tau+1}\right)$$

where

$$A = \frac{\log(1-\beta)+\frac{\beta}{1-\beta}\log(\beta)}{1-\beta}.$$

We thus arrived at equation (7), as stated in the lemma. □

A.2 Proof of Lemma 2

Proof. Using the definition of net worth, $n=(\bar{z}+p_t)k-Rb$, and replacing in the budget constraint of the bank (1), we obtain

$$c = \ n + q_t(b',k')b' - p_t k' \tag{A.7}$$

Updating the definition of net worth for the following period, we have

$$n' = (\bar{z}+p_{t+1})k' - (1+r)b'. \tag{A.8}$$

The value function under repayment can then be written with net worth as a single state variable with the law of motion given by (A.8).

We also have that $b' > \bar{b}_{t+1}(k')$ cannot be a choice of the bank because this would imply $q_t = 0$. As a result, the bank faces the risk-free price $q=1$ as long as $b' \leq \bar{b}_{t+1}(k')$.

Using (A.7), (A.8) and the equilibrium price and borrowing constraint, we arrive at (8). □

A.3 Proof of Lemma 3

Proof. **For part (i).** Let $n>0$ be the current net worth. Consider a policy such that $c=n>0$. Let $b'=p_t k'$ for some $k'>0$. This means that the budget constraint holds. Note that the borrowing constraint is:

$$b' \leq \gamma_t p_{t+1} k' \Leftrightarrow p_t \leq \gamma_t p_{t+1}$$

which is satisfied given the premise of part (i). Next period net worth is $n' = (\bar{z}+p_{t+1})k' - Rp_t k' = (R^k_{t+1}-R)p_t k'$ which is strictly positive and strictly increasing in k' given that $R^k_{t+1}>R$. Thus a bank can make its next period net worth arbitrarily large by having an arbitrarily large demand for capital. Given that the $\hat{V}^R_{t+1}(n') \geq V^D_t(k')$, and $V^D_t(k')$ goes to infinity as k' goes to infinity, it follows the bank valuation is infinite.

For part (ii). Note that from the budget constraint, together with the borrowing limit, we have

$$c = n + b' - p_t k' \leq n + (\gamma_t p_{t+1} - p_t)k'$$

And thus, given that $\gamma_t p_{t+1} < p_t$, a sufficiently large k' will generate a negative consumption. Thus, the demand for capital is finite.

Suppose now that the borrowing constraint is slack. That is $b' < \gamma_t p_{t+1} k'$. Consider now an increase in b' by $\Delta > 0$, small enough. with an associate increase in k' given by Δ / p_t. Note that this change leaves current consumption unchanged. In addition, $\Delta > 0$ can be chosen sufficiently small to keep the borrowing constraint holding. The change in net worth next period implied by this policy is given by $(R^k_{t+1} - R)\Delta > 0$, and thus we have found an improvement. It must be then that the borrowing constraint is binding.

For part (iii). Suppose that the demand for capital is strictly positive. Let (c, k', b') be a potential solution to the bank problem with $k' > 0$. Consider the following alternative policy with zero investment in capital: $(c, \tilde{k}', \tilde{b}') = (\tilde{c}, 0, b' - (\bar{z} + p_{t+1})k'/R)$. Using the law of motion for net worth, we can see that next-period net worth is given by

$$\tilde{n}' = (\bar{z} + p_{t+1})k' - Rb'$$

which is the same net worth as the original allocation. In addition, current consumption is higher with the new policy:

$$\tilde{c} = n + b' - \frac{p_t k'}{R} > n + b' - \ p_t k' = c$$

So the alternative policy delivers same continuation value and higher current consumption. Hence, an allocation with $k' > 0$ cannot be optimal. □

A.4 Proof of Lemma 4

We conjecture that the value function is

$$\hat{V}^R_t(n) = \ \frac{1}{1-\beta} \log(n) + \mathbb{B}^R_t \tag{A.9}$$

The borrowing constraint must be such that the bank does not default at $t+1$. That is,

$$\mathbb{B}^R_{t+1} + \frac{1}{1-\beta} \log(n') \geq \mathbb{B}^D_{t+1} + \frac{1}{1-\beta} \log((\underline{z} + p_{t+1})k')$$

Replacing n' for the law of motion and manipulating this expression, we arrive to

$$b' \leq \frac{\left[(\bar{z} + p_{t+1}) - (\underline{z} + p_{t+1}) e^{(1-\beta)(\mathbb{B}^D_{t+1} - \mathbb{B}^R_{t+1})}\right]}{R} k'$$

Therefore, the borrowing constraint takes a linear form, as conjectured. In particular,

$$b' \leq \gamma_t p_{t+1} k'$$

where γ_t is the leverage parameter and is given by

$$\gamma_t = \frac{(\bar{z} + p_{t+1}) - (\underline{z} + p_{t+1}) e^{(1-\beta)(\mathbb{B}^D_{t+1} - \mathbb{B}^R_{t+1})}}{R p_{t+1}}. \tag{A.10}$$

We establish now that if $R^k_{t+1} > R$, the borrowing constraint binds at time t.

Lemma A.1. *If $R^k_{t+1} > R$, then the bank is against the borrowing constraint.*

Proof. The proof is by contradiction. Denote $(c^*_t, k^*_{t+1}, b^*_{t+1})$ the solution to the bank problem with $b^*_{t+1} <$

$\gamma_t p_{t+1} k^*_{t+1}$. Consider the following alternative policy $(c^*_t, \tilde{k}_{t+1} + \Delta, \tilde{b}_{t+1} + \Delta p_t)$ with $0 < \Delta < \frac{\gamma_t p_{t+1} \tilde{k}_{t+1} - \tilde{b}_{t+1}}{p_t - \gamma_t p_{t+1}}$. The alternative allocation is feasible and delivers higher net worth since:

$$\begin{aligned}\tilde{n}_{t+1} &= (\tilde{k}_{t+1} + \Delta)(\bar{z} + p_{t+1}) - R\tilde{b}_{t+1} + \Delta p_t) \\ &= \tilde{k}_{t+1}(\bar{z} + p_{t+1}) - R\tilde{b}_{t+1}) + \Delta(R^k_{t+1} - R) \\ &> \tilde{k}_{t+1}(\bar{z} + p_{t+1}) - R\tilde{b}_{t+1} = n^*_{t+1}\end{aligned}$$

where $\tilde{n}_{t+1}$ and n^*_{t+1} are respectively the net worth under the alternative and original allocations.

Since the alternative allocation delivers the same consumption and higher net worth, this contradicts that the original allocation with a slack borrowing constraint is optimal. □

We now proceed to finish the proof of Lemma 4.

Proof. Consider first the case with $R^k_{t+1} > R$. From Lemma A.1, we know that borrowing constraint binds, and hence we can use $b' = \gamma_t p_{t+1} k'$. Replacing this in the law of motion for net worth and consumption, we obtain:

$$n' = k'(\bar{z} + p_{t+1}) - \gamma_t p_{t+1} k' R,$$

and

$$c = \; n - k'(p_t - \gamma_t p_{t+1}).$$

Replacing these two expressions and the conjectured value function (10) into (8), we have

$$\hat{V}^R_t(n) = \max_{k'} \log(n - k'(p_t - \gamma_t p_{t+1})) + \beta \left[\frac{1}{1-\beta} \log(k'(\bar{z} + p_{t+1}(1 - \gamma_t R)) + \mathbb{B}^R_{t+1} \right] \tag{A.11}$$

The first-order condition with respect to k' is

$$\frac{p_t - \gamma_t p_{t+1}}{n - k'(p_t - \gamma_t p_{t+1})} = \left(\frac{\beta}{1-\beta}\right) \frac{1}{k'}$$

and yield

$$k' = \frac{\beta n}{p_t - \gamma p_{t+1}}, \tag{A.12}$$

$$c = (1-\beta) n, \tag{A.13}$$

and

$$n' = \frac{\beta n}{p_t - \gamma_t p_{t+1}} (\bar{z} + p_{t+1}(1 - \gamma_t R))$$

Notice that by definition of R^e, we have that

$$R^e_{t+1} = \frac{\bar{z} + p_{t+1}(1 - \gamma_t R)}{p_t - \gamma_t p_{t+1}} \tag{A.14}$$

Using (A.12), (A.14), and replacing (A.9) on the left-hand side of (A.11)

$$\mathbb{B}_t^R + \frac{1}{1-\beta}\log(n) = \log\left((1-\beta)n\right) + \beta\left[\frac{1}{1-\beta}\log(\beta n R_{t+1}^e) + \mathbb{B}_{t+1}^R\right]$$

Rearranging this equation, we can observe that the terms multiplying $\log(n)$ cancel out. We therefore obtain that the conjecture is verified when the $\mathbb{B}_t^R$ satisfies:

$$\mathbb{B}_t^R = \frac{\beta}{1-\beta}\log(\beta) + \log(1-\beta) + \frac{\beta}{1-\beta}\log(R_{t+1}^e) + \beta\mathbb{B}_{t+1}^R \tag{A.15}$$

Iterating forward and imposing $\lim_{t\to\infty}\beta^t\mathbb{B}_t^R = 0$, we have

$$\mathbb{B}_t^R = \frac{1}{1-\beta}\left[\frac{\beta}{1-\beta}\log(\beta) + \log(1-\beta)\right] + \frac{\beta}{1-\beta}\sum_{\tau\geq t}\beta^{\tau-t}\log\left(R_{t+1}^e\right) \tag{A.16}$$

so the value under repayment is given by

$$V_t^R(n) = \frac{1}{1-\beta}\log(n) + \mathbb{B}_t^R$$

where $\mathbb{B}_t^R$ is given by (A.16). Equivalently, using definition of R^e and A, we arrive to

$$\hat{V}_t^R(n) = A + \frac{1}{1-\beta}\log(n) + \frac{\beta}{1-\beta}\sum_{\tau\geq t}^{\infty}\beta^{\tau-t}\log(R_{\tau+1}^e),$$

which is the expression (10).

Notice also from (A.12) and (A.13) and the fact that $b' = \gamma_t p_{t+1} k'$ that we have also verified the policies in item (ii) of the lemma for the case of $R_{t+1}^k > R$.

Finally, it is straightforward to verify that in the case of $R_{t+1}^k = R$, the conjectured value function (A.9) solves the Bellman equation and that the bank is now indifferent across b', k' while consumption remains given by (A.13). This completes the proofs of the three items in the lemma. □

A.5 Proof of Proposition 1

Proof. From the definition of γ_t in (A.10), we obtain

$$\frac{\beta}{1-\beta}\log\left(\frac{\bar{z} + p_{t+1}(1-\gamma_t R)}{\underline{z} + p_{t+1}}\right) = \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^R) \tag{A.17}$$

To obtain an expression for the right-hand side of (A.17), we use (A.6) and (A.15), and obtain that the difference in the intercepts in the value functions is given by

$$\mathbb{B}_t^D - \mathbb{B}_t^R = \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^R) + \frac{\beta}{1-\beta}\left[\log(R_{t+1}^D) - \log(R_{t+1}^e)\right]) \tag{A.18}$$

Using the definition of R^D and R^e and replacing (A.17)

$$\mathbb{B}_t^D - \mathbb{B}_t^R = \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^R) - \frac{\beta}{1-\beta}\left[\log\left(\frac{\bar{z} + p_{t+1}(1-\gamma_t R)}{p_t - \gamma_t p_{t+1}}\right) - \log\left(\frac{\underline{z} + p_{t+1}}{p_t}\right)\right]$$

Using that using that $\log(p_t - \gamma_t p_{t+1}) = \log\left(1 - \gamma_t \frac{p_{t+1}}{p_t}\right) + \log(p_t)$ and simplifying,

$$\mathbb{B}_t^D - \mathbb{B}_t^R = \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^R) - \frac{\beta}{1-\beta}\left[\log\left(\bar{z} + p_{t+1}(1-\gamma_t R)\right) - \log\left(1 - \gamma_t \frac{p_{t+1}}{p_t}\right) + \log(p_t) - \log\left(\frac{\underline{z} + p_{t+1}}{p_t}\right)\right]$$

Replacing (A.17) and simplifying, we arrive to

$$\mathbb{B}_t^D - \mathbb{B}_t^R = \frac{\beta}{1-\beta}\left[\log\left(1 - \gamma_t \frac{p_{t+1}}{p_t}\right)\right] \tag{A.19}$$

Updating (A.19) one period forward and replacing in (A.17):

$$\log\left(\frac{\bar{z} + p_{t+1}(1-\gamma_t R)}{\underline{z} + p_{t+1}}\right) = \beta \log\left(1 - \gamma_{t+1}\frac{p_{t+2}}{p_{t+1}}\right)$$

Simplifying we arrive

$$\frac{\bar{z} + p_{t+1}(1-\gamma_t R)}{\underline{z} + p_{t+1}} = \left(1 - \gamma_{t+1}\frac{p_{t+2}}{p_{t+1}}\right)^{\beta}$$

which is the expression in the proposition. □

A.6 Proof of Lemma 5

Proof. We have already argued that H is continuous, strictly increasing and strictly concave in $[0,1]$ and that $H(0) < 0$ and $H(1) \leq 1$.

Note that

$$H'(\gamma) = \frac{R}{R^D}\frac{1}{\beta}\left(\frac{R^k/R - \gamma}{R^D/R}\right)^{\frac{1-\beta}{\beta}}$$

Let γ_0 be such that $H'(\gamma_0) = 1$. This implies that

$$\gamma_0 = \frac{R^k}{R} - \frac{1}{\beta}\left(\frac{\beta R^D}{R}\right)^{\frac{1}{1-\beta}}$$

$$H(\gamma_0) = 1 - \left(\frac{\beta R^D}{R}\right)^{\frac{1}{1-\beta}}$$

Note that if $\beta R^D/R \geq 1$ then $R^k > R^D \geq R/\beta > R$ and thus $H(1) < 1$. Note that this implies that $H(\gamma_0) < 0$, and thus, together with concavity, it also implies that there is no fixed point in $[0,1]$.

For the case where $\beta R^D/R < 1$, we have that there are two solutions if $H(\gamma_0) > \gamma_0$. If $H(\gamma_0) = \gamma_0$, then there is just one fixed point. Finally if $H(\gamma_0) < \gamma_0$, then there are no solutions. This amount to checking the condition

$$\beta R^k/R < \beta + (1-\beta)(\beta R^D/R)^{\frac{1}{1-\beta}}$$

for two solutions, with equality for one, and with reverse inequality for none. □

A.7 Proof of Lemma 6

Part (i):

Proof. Based on Lemma 5, we first show that of the two fixed points of (11), one of them violates the no-Ponzi condition. For this is sufficient to check that only one of the two fixed points satisfies $\gamma < \gamma^{NP} = \frac{R - \beta R^k}{(1-\beta)R}$.

Note that it suffices then to show that $H(\gamma^{NP}) > \gamma^{NP}$, which is equivalent to:

$$1 - \left(\frac{R^k - R}{R^D(1-\beta)} \right)^{1/\beta} > \frac{R - \beta R^k}{R(1-\beta)}$$

If $R^k > R$, the inequality is equivalent to $\beta R^k / R < \beta + (1-\beta) \left(\beta R^D / R \right)^{\frac{1}{1-\beta}}$, and thus, the two fixed points lie at opposite sides of γ^{NP} and only the smaller one is valid.

If $R^k = R$, then $\gamma^{NP} = 1$ is a fixed point, and thus the other fixed point is necessary valid as it is less than γ^{NP}.

Let $\gamma^\star$ denote the valid fixed point. Note $\gamma^\star$ is the "unstable" solution to the dynamic system implied by $\gamma_{t+1} = H(\gamma_t)$. Thus if $\gamma_t < \gamma^\star$, then eventually the subsequent sequence of γ must become negative. On the other hand, if $\gamma_t > \gamma^\star$, then the subsequent sequence of γ converges to the highest fixed point from above, violating the no-Ponzi condition.

Thus the only equilibrium consistent sequence of borrowing limits keeps $\gamma_t = \gamma^\star$ at all times. □

Part (ii):

Proof. If $\beta R^D / R \geq 1$, or $\beta R^k / R > \beta + (1-\beta)(\beta R^D / R)^{1/(1-\beta)}$, then, from Lemma 5, there are no fixed points and $H(\gamma) < \gamma$ for all $\gamma \in [0, 1]$. This implies that any sequence of γ that satisfy $\gamma_{t+1} = H(\gamma_t)$ must eventually reach negative, a contradiction.

Note that if $\beta R^k / R = \beta + (1-\beta)(\beta R^D / R)^{1/(1-\beta)}$, there is unique fixed point, which corresponds exactly to γ^{NP}. Given that $H(\gamma) < \gamma$ for $\gamma < \gamma^{NP}$, this implies that any sequence where $\gamma_t < \gamma^{NP}$ for some t must eventually reach a negative value. In addition if $\gamma_t > \gamma^{NP}$, then the sequence converges to γ^{NP}, violating the no-Ponzi condition. □

A.8 Proof of Corollary 1

Proof. Note that the function $H(\gamma) = 1 - ((R^k - R\gamma)/R^D)^{1/\beta}$ is increasing in R, decreasing in R^k (and thus in $\bar{z}$), and increasing in R^D (and thus in $\underline{z}$). This immediately implies that the lowest fixed point is decreasing in R and $\underline{z}$ and increasing in $\bar{z}$.

For the comparative statics with respect to β note that H is decreasing in β for values of γ such that $H(\gamma) > 0$; the relevant domain range for the fixed points. It follows then that the lowest fixed point is increasing in β.

For the comparative statics with respect to p note that H can be written as $1 - \left(\frac{1 - \gamma R / R^k}{R^D / R^k} \right)^{1/\beta}$. An increase in p increases R^D / R^k and decreases R^k and thus increases H. Thus the lowest fixed point decreases with p. □

B Proofs for Sections 2.3– 2.5 (General Equilibrium)

B.1 Proof of Proposition 2

Part (i): Default equilibrium.

Proof. If all banks default, we have that the first-order condition for banks in equilibrium is

$$p^D = \beta(\underline{z} + p^D) \tag{B.1}$$

$$\Rightarrow p^D = \frac{\beta}{1-\beta}\underline{z} \tag{B.2}$$

Denoting by γ^D, the value of γ in a stationary equilibrium with default, we have, by (16) that

$$\gamma^D = H(\gamma^D, p^D) \tag{B.3}$$

To ensure existence of a default equilibrium, we must have a solution of H given the value of p^D. Note that by construction $\beta R^D = 1$ and thus $\beta R^D/R < 1$. Using the other condition in item (i) of Lemma 5 and replacing the value from p^D from (18), we arrive at the condition in the text. The fact that $\phi = 1$, $K^D_{t+1} = \bar{K}$, $K^R_{t+1} = 0$ and $B_{t+1} = 0$, $p_t = p^D$ and $\gamma_{t-1} = \gamma^D$ for all $t \geq 0$, $c = \underline{z}\bar{K}$ if $B_0 \geq \gamma^D p^D \bar{K}$ is immediate. □

Part (ii): Repayment equilibrium.

Proof. Taking first order conditions when the bank repays, we have that

$$\mu c = 1 - \beta(1+r) \tag{B.4}$$

$$\mu\gamma c = 1 - \beta\left(\frac{\bar{z}+p}{p}\right) \tag{B.5}$$

Combining these two we obtain an equation for p^R as a function of γ^R:

$$p^R = \frac{\beta\bar{z}}{1-\beta-(1-\beta R)\gamma^R}$$

Using the this, we have that a fixed point $\gamma = H(\gamma, p)$ requires that

$$(1-\gamma) = \left(\frac{\bar{z}(1-\gamma)}{(1-\beta)\underline{z} + \beta\bar{z} - (1-\beta R)\underline{z}\gamma}\right)^{1/\beta}$$

Ignoring the solution $\gamma = 1$ (which is never valid), we have that we are looking for a root of $h(\gamma)$:

$$h(\gamma) \equiv \bar{z}(1-\gamma)^{1-\beta} - [(1-\beta)\underline{z} + \beta\bar{z} - (1-\beta R)\underline{z}\gamma].$$

Note that $h(0) > 0$ and $h(1) < 0$, so h has a root in $(0,1)$. Note also that $h'(\gamma) = -(1-\beta)\bar{z}(1-\gamma)^{-\beta} + (1-\beta R)\underline{z}$ and that $h''(\gamma) < 0$ for $\gamma \in (0,1)$. Given that $h'(0) = -(1-\beta)(\bar{z}-\underline{z}) - \beta(R-1)\underline{z} < 0$ it follows that h is strictly decreasing in $(0,1)$ and thus has a unique root, γ^R.

Finally, note that

$$\gamma^{NP} = \frac{R-\beta R^k}{R(1-\beta)} = \gamma^R + (R-1)\frac{1-\gamma^R}{R(1-\beta)} > \gamma^R$$

and thus, the unique root $\gamma^R < \gamma^{NP}$ and is valid fixed point (it satisfies the no-Ponzi condition).

Starting from $B_0 = \gamma^R p^R \bar{K}$, this implies that it is an equilibrium that no banks default, $\phi = 0$ and the economy remains stationary at $p = p^D$. □

B.2 Proof of Proposition 3

Proof. We have that indifference at the stationary points imply

$$V^D((\underline{z} + p^D)\bar{K}; p^D) = V^R((\bar{z} + p^D)\bar{K} - R\bar{K}\gamma^D p^D; \{p^D, \gamma^D\})$$

and

$$V^D((\underline{z} + p^R)\bar{K}; p^R) = V^R((\bar{z} + p^R)\bar{K} - R\bar{K}\gamma^R p^R; \{p^R, \gamma^R\})$$

where we highlight the dependence of the stationary values on the equilibrium prices and borrowing limits.

In the stationary equilibrium with default, we have that defaulting banks choose to invest $\bar{K}$ and consume $c^D = \underline{z}\bar{K}$ forever. In the stationary repayment equilibrium, a bank that defaults could also choose to invest $\bar{K}$, consuming c^D forever. Thus, the value for a bank that defaults in the stationary repayment equilibrium must be weakly higher than in the default equilibrium:

$$V^D((\underline{z} + p^R)\bar{K}; p^R) \geq V^D((\underline{z} + p^D)\bar{K}; p^D)$$

This implies that the value of repayment in a stationary equilibrium in which banks repay must also be larger. That is, the three equations above imply that

$$V^R((\bar{z} + p^R)\bar{K} - RB^R, \{p^R; \gamma^R\}) \geq V^R((\bar{z} + p^D)\bar{K} - RB^D; \{p^D, \gamma^D\}) \tag{B.6}$$

Assume towards a contradiction of the Proposition that $\bar{B}^R = \gamma^R p^R > \gamma^D p^D = \bar{B}^D$. We can then show that (B.6) is violated.

In the stationary repayment equilibrium, consumption of a repaying bank is:

$$c_t^{RR} \equiv \bar{z}\bar{K} - (R-1)\bar{B}^R$$

for all t.

In the stationary default equilibrium, a repaying bank can achieve a policy of purchasing $\bar{K}$ every period, keep the same level of debt and consume c^{RD}:

$$c^{RD} \equiv \bar{z}\bar{K} - (R-1)\bar{B}^D > \bar{z}\bar{K} - (R-1)\bar{B}^R = c^{RR}$$

where the inequality follows from $\bar{B}^R > \bar{B}^D$.

Given that it is feasible for a repaying bank in a stationary default equilibrium to have higher consumption than a repaying bank in a stationary repayment equilibrium, it's value must be strictly higher. But then this contradicts (B.6). □

B.3 Proof of Proposition 4

The proof has several parts. Let us first state some preliminary results.

The evolution of K_t^D, the level of capital in defaulting banks, is as follows. Let N_t^D denote the net worth of defaulting banks, $N_t^D = (\underline{z} + p_t)K_t^D$. In equilibrium, $N_{t+1}^D = \beta((\underline{z} + p_{t+1})/p_t)N_t^D$. As a result, K_t^D evolves according to:

$$K_{t+1}^D = \beta \frac{\underline{z} + p_t}{p_t} K_t^D.$$

So, given a sequence of p_t and a initial value of K_0, we can determined the sequence of K_t^D for all $t \geq 1$. Note that this in equilibrium also determines the sequence of K_t^R as $\bar{K} = (1-\phi)K_t^R + \phi K_t^D$.

For repaying banks, let net worth be $N_t^R = (\bar{z} + p_t)K_t^R - RB_t$. Then, $N_{t+1}^R = \beta R_{t+1}^e N_t^R$. Thus, the evolution of B_t^R is

$$B_{t+1} = \frac{1}{R}\left[(\bar{z} + p_{t+1})K_{t+1}^R - \beta R_{t+1}^e \left((\bar{z} + p_t)K_t^R - RB_t\right)\right]$$

where recall R_{t+1}^e was defined in equation (9).

The sequence of prices and borrowing limits must also be consistent with the optimal capital decisions of repaying banks. If $R_{t+1}^k > R$ then

$$K_{t+1}^R = \frac{\beta((\bar{z} + p_t)K_t^R - RB_t)}{p_t - \gamma_t p_{t+1}}.$$

Otherwise, $p_{t+1} = Rp_t - \bar{z}$. Finally, equation (G) imposes a restriction on the evolution of γ_t and p_t.

We now derive the dynamic system. Let us define

$$\tilde{b}_t = \frac{(1-\phi)B_t}{\bar{K}}, \quad \tilde{k}_t = \frac{(1-\phi)K_t^R}{\bar{K}}, \quad \tilde{n}_t = \frac{(1-\phi)n_t}{\bar{K}}$$

Using that $K_{t+1}^D = \beta\frac{(\underline{z}+p_t)k}{p_t}$ from the bank problem under default and market clearing, (14), we arrive to

$$(1 - \tilde{k}_{t+1}) = \beta\left(\frac{\underline{z} + p_t}{p_t}\right)(1 - \tilde{k}_t) \tag{B.7}$$

Using the definitions, we also have

$$\tilde{n}_t = (\bar{z} + p_t)\tilde{k}_t - R\tilde{b}_t \tag{B.8}$$

From the bank's budget constraint:

$$p_t\tilde{k}_{t+1} - \tilde{b}_{t+1} = \beta\tilde{n}_t \tag{B.9}$$

Recall the equilibrium consistent borrowing limits are given by

$$\frac{\bar{z} + p_{t+1}(1 - \gamma_t R)}{\underline{z} + p_{t+1}} = \left(1 - \gamma_{t+1}\frac{p_{t+2}}{p_{t+1}}\right)^\beta \tag{G}$$

Note that above also holds at $t = -1$, as ϕ is interior.

Consider the left-hand side. Using that the borrowing constraint binds (as $R_{t+1}^k > R$ by the hypothesis of the proposition), and that $\tilde{b}_0 = \gamma_{-1}p_0k_0$, we obtain that

$$\frac{\bar{z} + p_{t+1}(1 - \gamma_t R)}{\underline{z} + p_{t+1}} = \frac{(\bar{z} + p_{t+1})\tilde{k}_{t+1} - R\tilde{b}_{t+1}}{(\underline{z} + p_{t+1})\tilde{k}_{t+1}} \tag{B.10}$$

for all $t \geq -1$.

Consider now the right-hand side of (G).

$$\left(1-\gamma_{t+1}\frac{p_{t+2}}{p_{t+1}}\right)^{\beta}=\left(\frac{\beta\tilde{n}_{t+1}}{p_{t+1}\tilde{k}_{t+2}}\right)^{\beta}$$
$$=\frac{(\beta\tilde{n}_{t+1})^{\beta}}{\left[p_{t+1}-\beta(\underline{z}+p_{t+1})(1-\tilde{k}_{t+1})\right]^{\beta}} \tag{B.11}$$

where the first line used (B.9) together with the binding borrowing constraint, and the second line used (B.7).

Combining (B.10) and (B.11), we obtain

$$\frac{\left[(\bar{z}+p_{t+1})\tilde{k}_{t+1}-R\tilde{b}_{t+1}\right]^{1-\beta}\left[p_{t+1}-\beta(\underline{z}+p_{t+1})(1-\tilde{k}_{t+1})\right]^{\beta}}{\beta^{\beta}(\underline{z}+p_{t+1})\tilde{k}_{t+1}}=1 \tag{B.12}$$

which is the expression in the proposition. Together with (B.7), (B.8), and (B.9) they conform the dynamic system.

To establish uniqueness, we first establish the following result

Lemma B.1. *In a mixed equilibrium with $\phi\in(0,1)$ and $R^{k}_{t+1}>R$ for all t, we have that (i) $R\tilde{b}_t>(\bar{z}-\underline{z})\tilde{k}_t$; and (ii) $p_t>\frac{\beta}{1-\beta}\underline{z}$ for all $t\geq 0$.*

Proof. **Part (i)** Suppose $R\tilde{b}_t\leq(\bar{z}-\underline{z})\tilde{k}_t$ for some $t\geq 0$. Then,

$$\tilde{n}_t=\tilde{k}_t(\bar{z}+p_t)-R\tilde{b}_t\geq(\bar{z}+p_t)\tilde{k}_t-(\bar{z}-\underline{z})\tilde{k}_t=(\underline{z}+p_t)\tilde{k}_t.$$

Hence a repaying bank at some point will have net-worth such that $N^R_t>(\underline{z}+p_t)K^R_t$. The fact that $R^e_{t+1}>R^D_{t+1}$, for all $t\geq 0$, implies that such a bank cannot be indifferent between default and repayment (and most strictly prefer to repay). Thus violating the binding borrowing constraint that requires indifference between default and repayment. Thus $R\tilde{b}_t>(\bar{z}-\underline{z})\tilde{k}_t$.

Part (ii) Suppose towards a contradiction that $p_t\leq\frac{\beta}{1-\beta}\underline{z}$.

Now consider (B.12). Summing and subtracting $\underline{z}k_t$, we obtain:

$$M\equiv\frac{\left[(\underline{z}+p_t)\tilde{k}_t-(R\tilde{b}_t-(\bar{z}-\underline{z})\tilde{k})\right]^{1-\beta}\left[p_t-\beta(\underline{z}+p_t)(1-\tilde{k}_t)\right]^{\beta}}{\beta^{\beta}(\underline{z}+p_t)\tilde{k}_t}=1$$

Using that $R\tilde{b}_t>(\bar{z}-\underline{z})\tilde{k}_t$ we have

$$M<\frac{\left[(\underline{z}+p_t)\tilde{k}_t\right]^{1-\beta}\left[p_t-\beta(\underline{z}+p_t)(1-\tilde{k}_t)\right]^{\beta}}{\beta^{\beta}(\underline{z}+p_t)\tilde{k}_t}$$
$$=\left[\left(\frac{p}{\beta(\underline{z}+p)}-1\right)\frac{1}{k}+1\right]^{\beta}\leq 1$$

where the last inequality follows from the fact that $\frac{p}{\beta(\underline{z}+p)}-1\leq 0$ if and only if $p_t\leq\frac{\beta}{1-\beta}\underline{z}$. We therefore reaching a contradiction that $M < 1$. We must have $p_t > \frac{\beta}{1-\beta}\underline{z}$. ◻

From (B.7), we have that

$$\tilde{k}_{t+1} = 1 - \beta\left(\frac{\underline{z} + p_t}{p_t}\right)(1 - \tilde{k}_t) \tag{B.13}$$

$$> 1 - (1 - \tilde{k}_t) = \tilde{k}_t \tag{B.14}$$

where the inequality follows from $p_t > \beta\frac{\underline{z}}{1-\beta}$. Using also that $k_0 = 1 - \phi$, we obtain $1 - (1-\phi)K_1^R/\bar{K} < \phi$, and $K_1^R > \bar{K}$. Market clearing then implies that $K_1^D < \bar{K}$. It follows then that $K_{t+1}^R > K_{t+1}^D$.

We now establish uniqueness of the dynamic evolution. That, we show that for any $\tilde{k}, \tilde{b}$ such that $R\tilde{b} > (\bar{z} - \underline{z})\tilde{k}$, there exists a unique value of p such that

$$M(p) \equiv \frac{\left[(\underline{z}+p)\tilde{k} - (R\tilde{b} - (\bar{z}-\underline{z})\tilde{k})\right]^{1-\beta}\left[p - \beta(\underline{z}+p)(1-\tilde{k})\right]^{\beta}}{\beta^{\beta}(\underline{z}+p)\tilde{k}} = 1$$

To see this note that

$$\lim_{p\to\infty} M(p) = \frac{\tilde{k}^{1-\beta}(1-\beta(1-\tilde{k}))^{\beta}}{k\beta^{\beta}} = \frac{1/\beta - 1 + \tilde{k}}{\tilde{k}} > 1 \tag{B.15}$$

In addition,

$$M\left(\frac{\beta\underline{z}}{1-\beta}\right) < \frac{((\underline{z}+p)k)^{1-\beta}}{(\underline{z}+p)k\beta^{\beta}}\left[\left(\frac{p}{\beta(\underline{z}+p)} - 1\right)\frac{1}{k} + 1\right]^{\beta}\Bigg|_{p=\beta\underline{z}/(1-\beta)} = 1 \tag{B.16}$$

So there exists a solution to $M(p) = 1$ with $p > \frac{\beta\underline{z}}{1-\beta}$.

For uniqueness, we have that $M'(p) > 0$ for $R\tilde{b} > (\bar{z} - \underline{z})\tilde{k}$ and $p > \beta\underline{z}/(1-\beta)$. Thus, there is a unique solution to $M(p) = 1$.

Finally, we show that $c_0^R < c_0^D$. Given the linear policy rules, it suffices then to show that $(\bar{z}+p_0)\bar{K} - RB_0 < (\underline{z}+p_0)\bar{K}$. But this follows immediately from $R\tilde{b}_0 > (\bar{z} - \underline{z})\tilde{k}_0$.

C Proofs for Section 3 (Bank Runs)

C.1 Proof of Lemma 7

Proof. Consider problem (22). We know, based on Lemma 4, that the continuation value can be expressed as

$$V_{t+1}^{Safe}(n) = \mathbb{B}_{t+1}^{Safe} + \frac{1}{1-\beta}\log(n) \tag{C.1}$$

where $\mathbb{B}_t^{Safe}$ has the same form as $\mathbb{B}_t^R$ from (A.15) but γ_t will be different as we will see.

$$\mathbb{B}_t^{Safe} = \frac{\beta}{1-\beta}\log(\beta) + \log(1-\beta) + \frac{\beta}{1-\beta}\log(R_{t+1}^e) + \beta\mathbb{B}_{t+1}^{Safe}$$

Replacing (C.1) into (22) and taking first-order conditions in (22), we obtain

$$\frac{1}{n - p_t k'} p_t = \frac{\beta}{1-\beta} \frac{1}{k'}$$
$$\Rightarrow k' = \beta \frac{n}{p_t} \tag{C.2}$$

Plugging back (C.2) into the right hand side of (22) and using (C.1), we obtain

$$\hat{V}_t^{Run}(n) = \log(n) + \log(1-\beta) + \beta \left[\frac{1}{1-\beta} \log\left(\beta R_{t+1}^k n\right) + \mathbb{B}_{t+1}^{Safe} \right]$$

After simplifying, we can express

$$\hat{V}_t^{Run}(n) = \mathbb{B}_t^{Run} + \frac{1}{1-\beta} \log(n) \tag{C.3}$$

where

$$\mathbb{B}_t^{Run} = \frac{\beta}{1-\beta} \log(\beta) + \log(1-\beta) + \frac{\beta}{1-\beta} \log\left(R_{t+1}^k\right) + \beta \mathbb{B}_{t+1}^{Safe} \tag{C.4}$$

Replacing the value for $\mathbb{B}_{t+1}^{Safe}$ from (C.1) in (C.4) and iterating forward, we obtain

$$\mathbb{B}_t^{Run} = A + \frac{\beta}{1-\beta} \left[\log\left(R_{t+1}^k\right) + \sum_{\tau \geq t+1} \beta^{\tau - t} \log(R_{\tau+1}^e) \right] \tag{C.5}$$

This completes the proof. □

C.2 Proof of Proposition 5

Proof. As in Proposition 1, we can use $\hat{V}_t^{Run}$ instead of $\hat{V}_t^R$ and obtain that:

$$\frac{\beta}{1-\beta} \log\left(\frac{\bar{z} + p_{t+1}(1 - \gamma_t R)}{\underline{z} + p_{t+1}} \right) = \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^{Run}) \tag{C.6}$$

To obtain an expression for the right-hand side of (C.6), we first use

$$\mathbb{B}_t^{Run} - \mathbb{B}_t^{Safe} = \frac{\beta}{1-\beta} \left[\log\left(R_{t+1}^k\right) - \log(R_{t+1}^e) \right]$$

Using (C.5) and (A.5)

$$\mathbb{B}_t^D - \mathbb{B}_t^{Run} = \frac{\beta}{1-\beta} \left[\log\left(R_{t+1}^D\right) - \log(R_{t+1}^k) \right] + \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^{Safe})$$

Adding and substracting $\beta \mathbb{B}_{t+1}^{Run}$, we get:

$$\mathbb{B}_t^D - \mathbb{B}_t^{Run} = \frac{\beta}{1-\beta} \left[\log\left(R_{t+1}^D\right) - \log(R_{t+1}^k) \right] + \beta(\mathbb{B}_{t+1}^D - \mathbb{B}_{t+1}^{Run}) + \beta(\mathbb{B}_{t+1}^{Run} - \mathbb{B}_{t+1}^{Safe}) \tag{C.7}$$

Combining (C.7) with (C.5) and (C.6):

$$\mathbb{B}_t^D - \mathbb{B}_t^{Run} = \frac{\beta}{1-\beta}\left[\log\left(R_{t+1}^D\right) - \log(R_{t+1}^k)\right] + \frac{\beta}{1-\beta}\log\left(\frac{\bar{z} + p_{t+1}(1-\gamma_t R)}{p_{t+1} + \underline{z}}\right) + \frac{\beta^2}{1-\beta}\left[\log\left(R_{t+1}^k\right) - \log(R_{t+1}^e)\right]$$

Updating one period forward and replacing in (C.6):

$$\frac{1}{1-\beta}\log\left(\frac{\bar{z} + p_{t+1}(1-\gamma_t R)}{p_{t+1} + \underline{z}}\right) = \frac{\beta}{1-\beta}\left[\log\left(R_{t+2}^D\right) - \log(R_{t+2}^k)\right] + \frac{\beta}{1-\beta}\log\left(\frac{\bar{z} + p_{t+2}(1-\gamma_t R)}{p_{t+2} + \underline{z}}\right) + \frac{\beta^2}{1-\beta}\left[\log\left(R_{t+2}^k\right) - \log(R_{t+2}^e)\right]$$

After algebraic manipulations, we arrive to the expression (G-run) in the proposition. □

C.3 Proof of Lemma 8

Proof. Recall that $\gamma^{NP} = \frac{R-\beta R^k}{R(1-\beta)}$, We have already argued that H^r is strictly concave in $\gamma \in [0,1]$. In addition, $H^r(0) = 1 - \left(\frac{R^k}{R^D}\right)^{1/\beta^2} < 0$ given that $R^k > R^D$ and $H^r(1) \leq 1$. Hence H^r admits at most two fixed points.

We are looking for a *stationary* value of γ such that $\gamma = H^r(\gamma, p)$ and $\gamma < \gamma^{NP}$.

For part (i). First, note that $\beta R^D/R < 1$ implies $\beta R^k/R < 1$. To see this, note that if $\beta R^k/R \geq 1$, then $\beta + (1-\beta)(\beta R^D/R)^{\frac{1}{1-\beta}}(\beta R^k/R)^{-\beta} < 1$, and thus the first condition in part (i) generates a contradiction.

Next, we have that $\beta R^k/R < 1$ implies that $\gamma^{NP} > 0$. In addition, that $R^k \geq R$ guarantees that $\gamma^{NP} \leq 1$.

The first condition in part (i) implies that $H^r(\gamma^{NP}) > \gamma^{NP}$. Thus, there are two fixed points in $(0,1]$, but only the lowest one is valid (that is, strictly less than γ^{NP}).

For part (ii). If $\beta R^D/R \geq 1$, then $\beta R^k/R > 1$ and $\gamma^{NP} \leq 0$. Thus any stationary solution in $(0,1)$ necessarily violates No Ponzi condition.

Suppose instead that $\beta R^D/R < 1$ and

$$\beta R^k/R \geq \beta + (1-\beta)(\beta R^D/R)^{\frac{1}{1-\beta}}(\beta R^k/R)^{-\beta} \tag{C.8}$$

Note that

$$H_r'(\gamma) = \left(1 + \frac{1-\beta}{\beta^2}\right)\left(1 - \frac{R}{R^k}\gamma\right)^{\frac{1-\beta}{\beta^2}}\left(\frac{R^k}{R^D}\right)^{\frac{1}{\beta^2}}\frac{R}{R^k}$$

Note that (C.8) implies that $H_r'(\gamma^{NP}) > 1$. To see this, suppose not and $H_r'(\gamma^{NP}) \leq 1$. Then, we have that

$$R^D \geq \left(\frac{1-\beta+\beta^2}{\beta}\right)^{\beta^2}(R^k)^{\beta(1-\beta)}\left(\frac{R}{\beta}\right)^{\beta^2}\left(\frac{R^k-R}{1-\beta}\right)^{1-\beta}$$

Given that $\frac{1-\beta+\beta^2}{\beta} \geq 1$, the above implies by

$$R^D > (R^k)^{\beta(1-\beta)} \left(\frac{R}{\beta}\right)^{\beta^2} \left(\frac{R^k - R}{1-\beta}\right)^{1-\beta}$$

But this is equivalent to $\beta R^k/R < \beta + (1-\beta)(\beta R^D/R)^{\frac{1}{1-\beta}}(\beta R^k/R)^{-\beta}$, a contradiction of (C.8). Thus a $H_r'(\gamma^{NP}) > 1$.

Given that $H^r(\gamma^{NP}) \leq \gamma^{NP}$ (from the same argument in part i) and $H_r'(\gamma^{NP}) > 1$, it follows that all potential fixed points are such that $\gamma \geq \gamma^{NP}$, a violation of the no-Ponzi condition. □

C.4 Proof of Proposition 6

Proof. The proof follows closely the proof of Proposition 2. Notice that, in fact, conditions (25) and (27) and are identical to (18) and (20).

For part (i): the default equilibrium.

The argument in the proof of Proposition 2 implies that $p^D = \beta \underline{z}/(1-\beta)$. And the value of γ^D must be a fixed point of H^r given p^D.

Note that $\beta R^D/R < 1$ as $\beta R^D = 1$ given the value of p^D. The condition that $\beta R^k/R < \beta + (1-\beta)(\beta R^D/R)^{\frac{1}{1-\beta}}(\beta R^k/R)^{-\beta}$ can be rewritten as requiring That

$$x < \beta + (1-\beta)\frac{x^{-\beta}}{R^{1/(1-\beta)}}$$

where $x \equiv \beta R^k/R$. This is equivalent to

$$x^{1+\beta} - \beta x^{\beta} - \frac{1-\beta}{R^{1/(1-\beta)}} < 0$$

The left hand side of the above inequality is strictly negative at $x = \beta$ and strictly positive at $x = 1$. In addition, h is convex for $x \in [\beta, \infty)$ and thus there is a unique value $x_0 \in [\beta, \infty)$ so that the left hand side is zero. This value is such that $x_0 \in (\beta, 1)$, and $x_0^{\beta}(x_0 - \beta) = (1-\beta)R^{-1/(1-\beta)}$.

For any value $x < x_0$, then we have that

$$\beta R^k/R < \beta + (1-\beta)(\beta R^D/R)^{\frac{1}{1-\beta}}(\beta R^k/R)^{-\beta}$$

and thus Lemma 8 implies that there is a valid stationary value of γ given p^D.

Rearranging the condition that $x < x_0$ we obtain the condition in part (i) of the Proposition.

For part (ii): the repayment equilibrium.

The same argument as in the proof of Proposition 2 delivers that the stationary price must solve (27) and that γ must be a fixed point of H^r given p.

Plugging the price into the fixd point equation, and manipulating (and ignoring the $\gamma = 1$ root, which cannot be valid), we have that γ^{rR} must solve:

$$h(\gamma) \equiv \bar{z}(1-\gamma)^{1-\beta}(1-(1-\beta R)\gamma)^{\beta(1-\beta)} - [(1-\beta)\underline{z} + \beta\bar{z} - (1-\beta R)\underline{z}\gamma] = 0$$

We note that $h(0) > 0$ and $h(1) < 0$. In addition h is strictly convex in $(0, 1)$, and thus it features a unique root.

The same argument as in the proof of Proposition 2 guarantees that such a root is strictly below the associated γ^{NP} given the corresponding equilibrium price, completing the proof. □

C.5 Proof of Lemma 9

Proof. **Part (i).** Given that $p^{rD} = p^D$ we dropped the dependence on the price in what follows.

We know that if a default equilibrium without runs exists, then

$$\frac{\bar{z}}{\underline{z}} < \frac{R-1}{\beta^{-1}-1} + R^{-\frac{\beta}{1-\beta}}.$$

But this implies that

$$\frac{\bar{z}}{\underline{z}} < \frac{R-1}{\beta^{-1}-1} + \frac{R^{-\frac{\beta}{1-\beta}}}{x_0^{\beta}}$$

as $x_0 \leq 1$. And thus, a default equilibrium with runs exists as well.

Consider now the following value of γ:

$$\gamma^w \equiv \frac{R^k}{R}\left[1 - \left(\frac{R^D}{R^k}\right)^{\frac{1}{1-\beta}}\right]$$

We have that $\gamma^D \geq \gamma^w$. To see this note that

$$H(\gamma^w) = 1 - \left(\frac{R^k - R\gamma^w}{R^D}\right)^{\frac{1}{\beta}} = 1 - \left(\frac{R^D}{R^k}\right)^{\frac{1}{1-\beta}} < \gamma^w$$

where the last inequality follows from $R^k > R^D = 1/\beta \geq R$. Note also that

$$H'(\gamma^w) = \frac{1}{\beta}\left(\frac{R^k - R\gamma^w}{R^D}\right)^{\frac{1-\beta}{\beta}} \frac{R}{R^D} = \frac{R^D}{R^k}R = \frac{R}{\beta R^k}$$

From the condition for the existence of a stationary equilibrium in Lemma 5, we know that $\beta R^k/R < 1$, $H'(\gamma^w) > 1$, and thus γ^w is a lower bound for the valid root γ^D as H is concave. That is, $\gamma^D > \gamma^w$.

Now consider

$$\begin{aligned}
H^r(\gamma^D) - \gamma^D &= (1-\gamma^D) + (R^k - R\gamma^D)^{1+\frac{1-\beta}{\beta^2}}(R^k)^{1/\beta-1}(R^D)^{-1/\beta^2} \\
&= (1-\gamma^D) - (R^D(1-\gamma^D)^{\beta})^{1+\frac{1-\beta}{\beta^2}}(R^k)^{1/\beta-1}(R^D)^{-1/\beta^2} \\
&= (1-\gamma^D)\left[1 - (1-\gamma^D)^{\frac{(1-\beta)^2}{\beta}}(R^k)^{\frac{1-\beta}{\beta}}(R^D)^{-\frac{1-\beta}{\beta}}\right] \\
&> (1-\gamma^D)\left[1 - (1-\gamma^w)^{\frac{(1-\beta)^2}{\beta}}(R^k)^{\frac{1-\beta}{\beta}}(R^D)^{-\frac{1-\beta}{\beta}}\right] > 0
\end{aligned}$$

where the second equality follows from $H(\gamma^D) = \gamma^D$, the first inequality from $\gamma^D > \gamma^w$, and the last inequality follows from the definition of γ^w and $R^k > R$.

The above implies that $H^r(\gamma^D) > \gamma^D$, and thus the smallest fixed point of H^r must be such that $\gamma^{rD} < \gamma^D$. This also implies that $\bar{B}^{rD} < \bar{B}^D$ as $p^D = p^{rD}$.

Part (ii). From the proof of Proposition 2, we have that γ^R is the unique solution to

$$h(\gamma) = \bar{z}(1-\gamma)^{1-\beta} - [(1-\beta)\underline{z} + \beta\bar{z} - (1-\beta R)\underline{z}\gamma] = 0$$

while from the proof of Proposition 6, γ^{rR} is the unique solution to

$$h_r(\gamma) \equiv \bar{z}(1-\gamma)^{1-\beta}(1-(1-\beta R)\gamma)^{\beta(1-\beta)} - [(1-\beta)\underline{z} + \beta\bar{z} - (1-\beta R)\underline{z}\gamma] = 0$$

For the case $\beta R = 1$, note that both functions are the same, and so are their unique roots, implying the same debt thresholds.

For the case $\beta R < 1$, note that $h_r(\gamma) < h(\gamma)$ for $\gamma \in (0,1)$. Note also that both functions cross zero from above, and thus, it follows that their unique roots are strictly ordered: $\gamma^{rR} < \gamma^R$. This implies that $p^{rR} < p^R$, as $\beta R < 1$ and the price is increasing in γ. That $\bar{B}^R < \bar{B}^{rR}$ follows from their respective definitions. □

C.6 Proof of Lemma 10

Proof. A defaulting bank in period 0 chooses $c_0^D = (1-\beta)(\underline{z} + p_0)\bar{K}$. A repaying bank facing a run optimally chooses $c_0^{Run} = (1-\beta)((\bar{z} + p_0)\bar{K} - RB_0)$. So it suffices to show that $\underline{z} + p_0 > (\bar{z} + p_0)\bar{K} - RB_0$ (the net worth under default is higher than under repayment facing a run). Suppose this were not the case. Then, it is feasible for a repaying bank facing a run to select the consumption and capital choices of the defaulting bank. This guarantees the first period flow utility for the repaying bank facing run is the same as that of the defaulting bank. Because $V_t^{Safe}(0,k) > V_t^D(k)$ for all $k > 0$, the continuation value for a repaying bank facing a run will be strictly higher than that of a defaulting bank. Thus, if $\underline{z} + p_0 \leq (\bar{z} + p_0)\bar{K} - RB_0$, the value of a repaying bank facing a run will be strictly higher than that of a defaulting bank, a contradiction of the interiority of ϕ.

Similarly, we know that the capital choices are $p_0 K_1^D = \beta(\underline{z} + p_0)\bar{K}$ and $p_0 K_1^{Run} = \beta((\bar{z} + p_0)\bar{K} - RB_0)$. The previous result that ranks the net worth also implies that $K_1^{Run} < K_1^D$. □

D Transitional dynamics

D.1 Case without bank runs: Convergence to the repayment equilibrium

In here we describe how the transition in the case of $B_0 < \bar{B}^R$ is obtained in the case without runs.

Recall that we consider in here the case of $\beta R < 1$. When debt is below $\bar{B}^R$, we conjecture that for T periods, the return to capital is exactly R, aggregate net worth decreases at rate βR, and the borrowing constraint does not bind. In period T, the borrowing constraint binds, the return to capital is higher than R, and the economy remains at the stationary repayment equilibrium thereafter.

To determine the value T, we use the following thresholds, which are defined recursively:

$$p^{\{T+1\}} = \frac{\bar{z}}{R} + \frac{1}{R}p^{\{T\}} \tag{D.1}$$

$$\bar{B}^{R,T+1} = \frac{1}{R}\left[(\bar{z} + p^{\{T+1\}}) - \frac{1}{\beta R}(\bar{z} + p^{\{T\}})\right]\bar{K} + \frac{1}{\beta R}\bar{B}^{R,T} \tag{D.2}$$

with initial conditions $p^{\{-1\}} = p^R$, and $\bar{B}^{R,-1} = \bar{B}^R$ as defined above. The idea behind the recursion above is that the return to capital equals R and the net worth decreases by a factor βR. This occurs up to the point where the economy hits the borrowing limit, $\bar{B}^R$, then the price equals p^R.

For any initial level of debt, B_0, we locate the T such that $B_0 \in [\bar{B}^{R,T}, \bar{B}^{R,T-1})$. A finite value $T \geq 0$ exists for any initial debt $B_0 < \bar{B}^R$. Using this value of T, we obtain the initial price of capital, p_0, by solving the following system:

$$B_T = \frac{1}{R}\left[(\bar{z}+p_T)\bar{K} - (\beta R)^T\left((\bar{z}+p_0)\bar{K} - RB_0\right)\right] \tag{D.3}$$

$$p_0 = \sum_{j=1}^{T}\frac{\bar{z}}{R^j} + \frac{p_T}{R^T} \tag{D.4}$$

$$p_T = p^R + \frac{\beta R(\bar{B}^R - B_T)/\bar{K}}{1-\beta}, \tag{D.5}$$

where p_T and B_T represent, respectively, the price and aggregate debt level in period T, where T is the period right before the economy transitions to the stationary state.

The price of capital in period T, p_T, must guarantee that the aggregate demand for capital equals the supply $\bar{K}$. Note that aggregate net worth in this period is $N_T = (\bar{z}+p_T)\bar{K} - RB_T$. Using the conjecture that $R_T^k > R$, the demand for capital from Proposition 4 is $\beta N_T/(p_T - \gamma^R p^R)$. Market clearing in this period then implies

$$\frac{\beta[(\bar{z}+p_T)\bar{K} - RB_T]}{p_T - \gamma^R p^R} = \bar{K},$$

which delivers, using the definition of $\bar{B}^R$ and the value of p^R in the stationary repayment equilibrium, equation (D.5). Using the conjectured evolution of net worth delivers equation (D.3). And finally, using the conjectured return equal to R for the first T periods delivers (D.4). Our threshold definition guarantees that $\bar{B}_T \in [\bar{B}^{R,0}, \bar{B}^R)$ and that p_T is such that $(\bar{z}+p_T)/p_T \geq R$.

Having obtained an initial price p_0, we can determine p_t for all $t < T$, using that the capital return is R. The sequence of $\{B_t\}$ can then be obtained using that

$$B_t = \frac{1}{R}\left[(\bar{z}+p_t) - \frac{1}{\beta R}(\bar{z}+p_{t+1})\right]\bar{K} + \frac{1}{\beta R}B_{t+1}, \text{ for any } t < T. \tag{D.6}$$

Finally, we can obtain the associated γ_t for $t \in \{-1, 0, ..T\}$ using equation (G), given the sequence of prices and the terminal value of $\gamma_T = \gamma^R$.

D.2 Case with runs: Convergence to the repayment equilibrium

The value of T is determined in the same way as in the case without runs. That is, we use equations (D.1) and (D.2) but with initial conditions $p^{-1} = p^{rR}$ and $\bar{B}^{R,-1} = \bar{B}^{rR}$. With these thresholds, we can locate the value of T such that $B_0 \in [\bar{B}^{R,T}, \bar{B}^{R,T-1})$. Given this value of T, we solve the system (D.3), (D.4), and (D.5), which solves for the initial price p_0 and the price at T, p_T. We can then use that the capital return equals R for all $t < T$ to obtain all the prices for all $t \in \{1, \ldots, T-1\}$. The sequence of aggregate debt levels is then obtained using equation (D.6). Finally, using that $\gamma_t = \gamma^{rR}$ for all $t \geq T$, we can then use equation (G-run) to obtain the sequence of γ_t for $t < T$.