International Reserve Management under Rollover Crises*

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Abstract

This paper investigates how a government should manage international reserves when it faces the risk of a rollover crisis. We ask, should the government accumulate reserves or reduce debt to make itself less vulnerable? We show that the optimal policy entails initially reducing debt, followed by a subsequent increase in both debt and reserves as the government approaches a safe zone. Furthermore, we find that issuing additional debt to accumulate reserves can lead to a reduction in sovereign spreads. Evidence from a panel of emerging economies is consistent with these predictions: increases in reserves financed by public external borrowing are associated with lower spreads, and reserve holdings are not systematically drawn down during crisis episodes.

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1 Introduction

Governments are frequently exposed to episodes where investors suddenly lose confidence in the government's ability to repay its debt. This sudden loss of confidence can be self-fulfilling and result in situations where the government is unable to roll over the debt and defaults on its obligations. Given the significant costs associated with rollover crises, understanding how governments can reduce their vulnerability is crucial for economic policy.

In this paper, we investigate whether governments should accumulate international reserves as a safeguard against rollover crises. A common argument is that reserves provide the government with liquid resources when it cannot roll over its debt. However, a government can also lower its vulnerability by reducing sovereign debt. To the extent that reserves earn a lower interest than the one paid on the debt, it is unclear whether building up a stock of reserves is an effective way to reduce the government's vulnerability to a rollover crisis.

We provide a simple environment to study the optimal management of international reserves for a government subject to the risk of rollover crises. We show that if the government is highly indebted, accumulating reserves is not optimal. Rather, the government should lower its vulnerability by reducing the debt. We show that once the debt is reduced significantly, accumulating reserves becomes strictly optimal. In particular, the government should finance the reserve accumulation by increasing the amount of borrowing. Notably, issuing more debt in this case *reduces* sovereign spreads.

These findings highlight an alternative perspective on international reserves: reserves are valuable here not because they serve as a buffer when the government faces adverse shocks, but because they act as a *liquidity backstop* that prevents self-fulfilling rollover crises. Consistent with the mechanism in our model, we find that increases in reserves financed by public external borrowing are associated with lower spreads, and reserve stocks are not systematically drawn down during crisis episodes. By contrast, the standard buffer-stock view holds that reserves are accumulated as insurance and drawn down in bad times—and that this insurance should raise spreads, since larger gross positions increase borrowing costs.

Our environment is a canonical model of rollover crises following Cole and Kehoe (2000), augmented with reserve accumulation. The government starts with an initial stock of reserves and long-term debt, receives a constant stream of income, and discounts the future at the same rate as external investors. To abstract from the insurance motive highlighted in Bianchi, Hatchondo and Martinez (2018), we assume that the only source of uncertainty is the possibility of a rollover crisis.[1] When the government is in good credit standing, it decides how much debt to issue, what level of reserves it should accumulate, and whether to repay the coupons due or default on its obligations. Upon default, the government faces a penalty and is permanently excluded from sovereign debt markets. However, it can keep its reserves and continue to accumulate reserves in the future.

Our analysis begins by characterizing how the economy can be in one of three zones depending on the initial portfolio of debt and reserves. In the safe zone, the government repays the debt regardless of whether investors continue to roll over the debt, and thus it is not vulnerable to a run in equilibrium. In the default zone, the government finds it optimal to default regardless of whether investors are willing to lend. In the crisis zone, the government's default decision depends on investors' beliefs. If investors are willing to roll over the debt, it is optimal for the government to repay. On the other hand, if investors refuse to roll over the debt, the cost of repayment increases for the government, leading it to default. In the crisis zone, the government is therefore vulnerable to self-fulfilling rollover crises, as in Cole and Kehoe (2000).

The key question we tackle is the following: Suppose a government is in the crisis zone. What is the best strategy to reach the safe zone? Should the government accumulate reserves or reduce its debt?

A key consideration for understanding our results is how gross positions affect the value of both default and repayment for the government. An initial point to note is that higher reserves increase both the value of repayment and the value of default. In contrast, higher debt reduces the value of repayment and does not affect the value of default. In an environment with one-period debt, an increase of one unit in both debt and reserves leaves the value of repayment for the government unchanged. Because the value of default is increasing in reserves, an increase in debt and reserves makes the government more vulnerable to a rollover crisis by making default more attractive. In this scenario, with one-period debt, accumulating reserves is therefore not optimal. The government should reduce its vulnerability to a rollover crisis by reducing debt.

When debt has a long maturity, however, larger gross positions—holding net foreign assets constant—allow the government to relax its budget constraint during a run, because only a fraction of the debt comes due each period. For a given net foreign asset position, a joint increase in debt and reserves therefore increases the resources available to the government and raises the value of repayment. As mentioned above, however, the value of default also

[1] We also abstract from maturity management as an alternative way to reduce the vulnerability (see Aguiar and Amador, 2013).

increases with higher gross positions. Therefore, the overall effect of higher reserves and debt on the exposure to a rollover crisis is, in principle, ambiguous. Our analytical results elucidate when it is optimal to simply reduce the debt to reduce the vulnerability and when it is optimal to accumulate reserves.

Moreover, for a highly indebted government, the transition out of the crisis zone is *non-monotonic*: the optimal policy entails initially reducing debt, followed by an increase in both debt and reserves as the government exits the crisis zone. That is, in the initial phase, the government should raise its net foreign asset position through reductions in debt and run down any initial reserves to zero, postponing reserve accumulation until reserve holdings are sufficient to make the government safe from a rollover crisis. Importantly, upon exiting, the government issues more debt to accumulate reserves, and this operation lowers sovereign spreads.

Finally, we present two empirical patterns that are consistent with the rationale for reserves in our model. First, in a panel of emerging economies, reserve accumulation financed by public external borrowing is associated with lower spreads. Second, during crisis episodes, reserves are typically not drawn down. Together, these facts support the model's implication that reserves function as a liquidity backstop that deters rollover crises: by reducing investors' incentives to run, reserves lower default risk and sovereign spreads. This contrasts with the standard buffer-stock view, which predicts that reserves are accumulated as insurance to be depleted in crises and that larger gross positions raise spreads.

Related literature. Our paper belongs to the literature on international reserves. In particular, our paper is related to studies on the joint determination of reserves and defaultable sovereign debt that build on the workhorse model of sovereign default (Eaton and Gersovitz, 1981; Aguiar and Gopinath, 2006; and Arellano, 2008).[2] The key new perspective in our paper is that international reserves are not accumulated as insurance—accumulated in good times and drawn down during crises—but rather to prevent self-fulfilling rollover crises.

Alfaro and Kanczuk (2009) study reserve accumulation in the canonical sovereign default model with one-period debt and show that it is not optimal for the government to accumulate reserves. This is because reserves make default more attractive, thereby worsening debt sustainability. In a model with long-term debt, Bianchi, Hatchondo and Martinez (2018) show that accumulating reserves provides insurance against negative income shocks that raise borrowing costs. In their model, when the government issues debt to accumulate reserves, it

[2]See Aguiar and Amador (2014) and Aguiar, Chatterjee, Cole and Stangebye (2016) for a review of the literature on sovereign debt and default.

effectively reallocates resources from states with high bond prices and low marginal utility to states with low bond prices and high marginal utility. On the other hand, higher gross positions lead to higher sovereign spreads. The optimal portfolio for the government trades off the insurance benefits against the costs of higher spreads.[3] In contrast to these studies, we consider the possibility of rollover crises and show that issuing debt to accumulate reserves can reduce the vulnerability to rollover crises and lead to a *decrease in sovereign spreads*. Hernandez (2018) provides numerical simulations in a model with fundamental risk and rollover risk. Our approach focuses exclusively on rollover risk and allows us to provide an analytical characterization of when issuing debt to accumulate reserves is optimal.[4]

Corsetti and Maeng (2023) study the joint accumulation of reserves and debt in the model of belief-driven sovereign risk crises proposed by Aguiar, Chatterjee, Cole and Stangebye (2022). They show that accumulating reserves and one-period debt is desirable because it rules out an inefficient equilibrium with depressed bond prices.[5] The distinct role of reserves in our model is that they provide the government with a liquidity backstop in the event that investors refuse to roll over the debt in the future.

Aguiar and Amador (2013, 2025) and Bocola and Dovis (2019) study the optimal debt maturity structure in models with rollover crises. In these studies, a sufficiently long maturity can prevent a rollover crisis, but it has the cost of exacerbating debt dilution.[6] Our results on the optimal deleveraging are connected in particular to those in Aguiar and Amador (2013), which show that it is optimal for the government to initially remain passive in long-term debt bonds—actively reducing short-term debt—and to lengthen the maturity at the end of the process. Aguiar and Amador (2025) study maturity swaps and show how they can improve bond prices and welfare under rollover risk. Our paper considers an exogenous maturity structure and studies the optimal accumulation of international reserves. One difference

[3]Bianchi and Sosa-Padilla (2024) show that under a fixed exchange rate regime, the hedging benefits lead to a macroeconomic stabilization motive for accumulating reserves.

[4]In his simulations, Hernandez (2018) finds negligible effects of changes in rollover risk on the accumulation of reserves, suggesting that the key motive for reserve accumulation in his model is related to fundamental risk instead of rollover risk. Our necessary condition for reserves to be optimal under a rollover crisis does not appear to be satisfied in his calibration.

[5]Aguiar et al. (2022) show how the equilibrium with depressed bond prices can also be avoided by issuing low or high amounts of debt, relative to the 'good' equilibrium level. Central to the mechanism in their framework is that intra-period uncertainty about the relative payoff between default and repayment induces a price schedule that is a non-monotonic function of debt.

[6]The desirable properties of long-term debt are also highlighted in Cole and Kehoe (2000). See also Arellano and Ramanarayanan (2012) and Hatchondo, Martinez and Sosa-Padilla (2016) for studies of the maturity tradeoffs under fundamental risk. There is also an extensive literature on maturity tradeoffs in closed economies (Barro, 1999, 2003; Angeletos, 2002; Buera and Nicolini, 2004; Bhandari, Evans, Golosov and Sargent, 2017), in which the key aspects involve distortionary taxation and fluctuations in the endogenous risk-free rate instead of default risk.

in our portfolio problem is that reserves increase the value of the outside option (default), whereas the outside option of defaulting is exogenous in their model. The fact that reserves increase the value of default is important because it can potentially reduce the sustainable debt level, as in Bulow and Rogoff (1989).[7]

Conesa and Kehoe (2024) study a model of rollover crises where the government can commit to a tax rate in advance. They show that it is ex-ante optimal to set a high tax so that in the event of a rollover crisis, the government has sufficient tax revenue to repay the debt, thus deterring investors from running. However, this policy is suboptimal ex-post, as the government can borrow at the risk-free rate and would prefer to set a lower tax rate. They refer to this policy as "preemptive austerity."[8] In our model, reserves also serve a preemptive role, but using reserves to service the debt is ex-post optimal and does not require ex-ante commitment. Moreover, the accumulation of reserves allows the government to maintain a lower net foreign asset position and potentially higher consumption as it exits the crisis zone.

2 Environment

Consider a small open economy with time indexed by $t = 0, 1, 2, \ldots$. There is a single consumption good, which is freely tradable. The government in the small open economy receives a constant endowment of the consumption good, y, and trades bonds, b, and risk-free assets, a, with a continuum of external investors. Investors are risk-neutral and share the same discount factor as the government.

2.1 Government Problem

The government's preferences over consumption streams are represented by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $\mathbb{E}$ is an expectation operator. We assume that $\beta \in (0, 1)$ and $u(\cdot)$ is an increasing, twice-continuously differentiable and strictly concave function defined over non-negative

[7] Bulow and Rogoff (1989) show that lending cannot be supported in equilibrium in the absence of direct punishments when the government has the ability to save abroad in reserves.

[8] See also a related literature on sovereign debt crises that followed the Eurozone crisis (e.g., Aguiar, Amador, Farhi and Gopinath, 2013, 2015; Bacchetta, Perazzi and Van Wincoop, 2018; Corsetti and Dedola, 2016; Corsetti and Maeng, 2024; Araujo, Leon and Santos, 2013; Lorenzoni and Werning, 2019; Ayres, Navarro, Nicolini and Teles, 2018, 2023; Camous and Cooper, 2019; Bassetto and Galli, 2019; Cole, Neuhann and Ordonez, 2025; Bianchi and Mondragon, 2022; Bianchi, Ottonello and Presno, 2023).

values, and satisfies the Inada conditions $\lim_{c\to 0^+} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. We denote $\underline{u} = u(0)$ and $\lim_{c\to\infty} u(c) = \bar{u}$ with $\underline{u}, \bar{u} \in (-\infty, \infty)$.

The government enters every period with an initial portfolio of assets and outstanding bonds, (a, b). We refer to the assets as "reserves" and assume without loss of generality that they have a one-period maturity.[9] The return on reserves is $1 + r = \beta^{-1}$.

The bonds the government issues are long-maturity bonds with geometrically decaying coupons, as in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). A bond issued at any date t promises to pay $\left(\frac{\delta+r}{1+r}\right)[1, (1-\delta), (1-\delta)^2, ...]$ in periods $t+1, t+2, t+3....$ Notice that the coupon payments are normalized so that the price of the bond equals $1/(1+r)$ if there is no default risk. The Macaulay duration of the bond is parameterized by $1/\delta$. We highlight that we treat debt duration as given in order to focus on the management of international reserves.[10]

If the government has not defaulted in the past, we can write its budget constraint when it chooses to repay in period t as

$$c_t = y + a_t - \left(\frac{\delta + r}{1 + r}\right) b_t - \frac{a_{t+1}}{1 + r} + q_t[b_{t+1} - (1 - \delta)b_t].$$

That is, the government collects its income and assets, consumes, pays the debt coupons, accumulates reserves, and issues bonds at a price schedule q_t. As we will see, in the Markov equilibrium, the bond price schedule will depend on the portfolio chosen by the government.

If the government defaults on its debt, it is excluded permanently from financial markets and faces a utility cost ϕ every period.[11] As occurs in practice, the government is able to keep its holdings of reserves and continue to adjust them over time.[12] The budget constraint in case of default is as follows:

$$c_t = y + a_t - \frac{a_{t+1}}{1 + r}\,.$$

Timing. Following Cole and Kehoe (2000), we assume that the government chooses its portfolio—new debt issuance and reserve accumulation—at the beginning of the period, and makes the repayment/default decision at the end of the period. As highlighted in Aguiar and Amador (2014), this timing contrasts with the Eaton-Gersovitz setup, in which the

[9] When reserves can be sold costlessly in a spot market and there are no fluctuations in the risk-free interest rate, it is equivalent whether reserves are one-period or long-term assets. The equivalence would break down in the presence of shocks to the risk-free rate.

[10] Optimal maturity management is analyzed in Aguiar and Amador (2013).

[11] Our results about the optimal portfolio are qualitatively the same if we impose an income cost of defaulting.

[12] Under the Foreign Sovereign Immunities Act (FSIA), reserves cannot be legally seized by creditors.

government first decides whether to repay or default and, conditional on repayment, selects its portfolio within the period.[13] We also assume that if the government chooses to default at the end of the period, it can reoptimize the level of reserves. As in the Cole–Kehoe model, the fact that investors are atomistic will open the door to coordination failures where a *good equilibrium*, in which investors continue to roll over the bonds and the government repays, coexists with a *bad equilibrium*, where investors refuse to roll over the bonds and the government defaults.

We will use ζ to denote a sunspot variable that will determine the type of equilibrium. If $\zeta = 0$, investors will expect others to continue rolling over the bonds, while if $\zeta = 1$, investors will expect others to stop rolling over the bonds. The probability that $\zeta = 1$ is constant and denoted by λ. As we will see, whether a rollover crisis actually takes place will be determined endogenously and will depend on the initial portfolio of the government.

Recursive problem. If the government has not defaulted in the past, it chooses whether to repay or default, and its value function is given by

$$V(a, b, \zeta) = \max\left\{V_R(a, b, \zeta), V_D(a)\right\},$$

where V_D and V_R represent, respectively, the values of default and repayment. We assume without loss of generality that if the government is indifferent between repaying and defaulting, it repays.

The value under default is given by

$$V_D(a) = \max_{c \geq 0, a' \geq 0} \left\{u(c) - \phi + \beta V_D(a')\right\}, \tag{1}$$

$$\text{subject to}$$

$$c = y + a - \frac{a'}{1+r}.$$

[13] Following the formulation in Aguiar and Amador (2013), we assume that if the sovereign defaults at settlement, the auction proceeds are distributed pro rata across all bondholders according to their respective positions, as captured by the budget constraint under default. As they note, this differs slightly from Cole and Kehoe (2000), where the sovereign retains the proceeds from newly issued bonds. The Aguiar and Amador specification is convenient because it avoids having to track off-equilibrium outcomes in which the sovereign issues an arbitrarily large amount of new debt, consumes the proceeds, and then defaults. As in Cole and Kehoe (2000), however, the key assumption is that newly issued bonds are immediately exposed to default risk.

The value of repayment is given by

$$V_R(a,b,\zeta) = \max_{c\geq 0, a'\geq 0, b'} \{u(c) + \beta\, \mathbb{E} V(a',b',\zeta')\},$$

$$\text{subject to}$$

$$c = y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + Q(a',b',s)\,(b' - (1-\delta)b)\,,$$

where we index the bond price schedule by the initial state $s = (a,b,\zeta)$ to capture the possibility of multiple equilibria, as will become clear below. If the constraint set is empty, we set the value to $\underline{u}/(1-\beta)$. [14]

2.2 Markov Equilibrium

Given investors' risk neutrality, non-arbitrage implies that the expected return from government bonds has to be equal to the risk-free rate. That is,

$$Q(a',b',s) = \begin{cases} \frac{1}{1+r}\mathbb{E}\left[(1-d')\left(\frac{\delta+r}{1+r} + (1-\delta)Q(a'',b'',s')\right)\right] & \text{if } d(s) = 0, \\ 0 & \text{if } d(s) = 1, \end{cases} \tag{2}$$

where $b'' = b'(s')$, $a'' = a'(s')$, and $d' = d(s')$ represent the policies the government is expected to follow in the next period, and $d_t = 0$ if the government repays and 1 otherwise. If the government is expected to repay at the end of the current period, the bond price equals the discounted expected value of next period's coupon payment plus the bond's expected continuation value (its next-period secondary-market price). If the government is expected to default at the end of the current period, the bond price is zero.

The Markov equilibrium in this economy is defined as follows:

Definition 1. A Markov equilibrium is defined by a value function V and associated policies $\{c(\cdot), d(\cdot), a'(\cdot), b'(\cdot)\}$, and a bond price schedule $Q(\cdot)$ such that

i) the policies and the value function solve the government problem given the bond price schedule;

ii) the price schedule satisfies (2) given the government policies.

[14] We also restrict Ponzi schemes by assuming that debt cannot exceed the present value of the endowment, $y(1+r)/r$.

2.3 Multiplicity of Equilibria

As in Cole and Kehoe (2000), the government may be subject to rollover crises, in which it defaults because investors stop rolling over its bonds. To determine the states in which the government is vulnerable to a rollover crisis, we need to distinguish between two scenarios: one in which investors continue to roll over the bonds and one in which they do not.

Consider first a situation in which each individual investor expects others to continue rolling over the bonds. In this case, the government solves the following problem:

$$V_R^+(a,b) = \max_{c\geq 0, a'\geq 0, b'} \{u(c) + \beta\,\mathbb{E}V(a',b',\zeta')\}, \tag{3}$$
$$\text{subject to}$$
$$c = y + a - \left(\frac{\delta+r}{1+r}\right)b - \frac{a'}{1+r} + q(a',b')\,(b' - (1-\delta)b)\,,$$

where q denotes the "fundamental" bond price:

$$q(a',b') = \frac{1}{1+r}\mathbb{E}\left\{(1-d')\left[\left(\frac{\delta+r}{1+r}\right) + (1-\delta)\,Q(a'',b'',s')\right]\right\}. \tag{4}$$

Consider now a situation in which the government would like to issue new debt, $b' > (1-\delta)b$, but investors are unwilling to lend. In this case, the value of repayment for the government is given by

$$V_R^-(a,b) = \max_{c\geq 0, a'\geq 0, b'} \{u(c) + \beta\,\mathbb{E}V(a',b',\zeta')\}, \tag{5}$$
$$\text{subject to}$$
$$c = y + a - \left(\frac{\delta+r}{1+r}\right)b - \frac{a'}{1+r} + q(a',b')\,(b' - (1-\delta)b)\,,$$
$$b' \leq (1-\delta)b\,.$$

An immediate implication is that $V_R^+(a,b) \geq V_R^-(a,b)$. That is, a government that can roll over its debt obtains at least the same value as a government that cannot.[15,16]

When $V_R^-(a,b) < V_D(a) \leq V_R^+(a,b)$, multiple equilibria arise. If the government faces the fundamental price q, it repays its debt; if instead it faces a price $q = 0$, it defaults. We

[15] One element implicit in the budget constraint in problem (5) is that if the government were to repurchase debt, by choosing $b' < (1-\delta)b$, when investors are unwilling to lend, the price of bonds would rise to the fundamental price (see Aguiar and Amador, 2014).

[16] If the government does not wish to borrow in the first place, then the value of repaying is unaffected by a potential run.

characterize below the set of initial portfolios for which such self-fulfilling debt crises can occur.

The safe zone, the crisis zone, and the default zone. Given the value functions (1), (3), and (5), we can split the economy into three different zones depending on the initial portfolio (a, b):

$$\mathbf{S} = \Big\{(a, b) : V_D(a) \leq V_R^-(a, b)\Big\},$$
$$\mathbf{D} = \Big\{(a, b) : V_D(a) > V_R^+(a, b)\Big\},$$
$$\mathbf{C} = \Big\{(a, b) : V_R^-(a, b) < V_D(a) \leq V_R^+(a, b)\Big\}.$$

S is the safe zone: the government is better off repaying, regardless of the sunspot realization. **D** is the default zone: the government is better off defaulting, irrespective of the sunspot realization. **C** is the crisis zone: the government finds it optimal to repay when lenders are willing to lend, whereas it finds it optimal to default if the lenders are unwilling to lend.

Debt thresholds. We now construct debt thresholds that determine the boundary between **S** and **C**, and the boundary between **C** and **D**, for every a. We make the following assumption:

Assumption 1. *The default cost ϕ is such that there exists $u_{min} > \underline{u}$ satisfying*

$$u_{min} + \beta\frac{\bar{u}}{1-\beta} \leq \frac{u(y) - \phi}{1-\beta}.$$

As in Aguiar, Amador, Hopenhayn and Werning (2019), this assumption ensures that receiving zero consumption triggers default, and simplifies the characterization. Using that the value functions of repayment are decreasing in debt and increasing in reserves, and continuous in both, we have the following lemma.[17]

Lemma 1. *For every a, there exists unique finite thresholds $b^+(a)$ and $b^-(a)$ such that*

$$V_R^-(a, b^-(a)) = V_D(a), \tag{6}$$

and

$$V_R^+(a, b^+(a)) = V_D(a). \tag{7}$$

[17] Monotonicity and continuity are shown in Auxiliary Lemmas B.1 and B.2.

Moreover, $b^+(a) \geq b^-(a)$.

Proof. In Appendix A.1. □

We have constructed two functions that map an initial level of reserves to a level of debt that makes the government indifferent between repaying and defaulting, depending on whether investors are willing to roll over the debt or not. A central aspect of our analysis will be how these thresholds vary with reserves and what this implies for the optimal portfolio as the government tries to exit the crisis zone.

2.4 Equilibrium Payoffs

Given the definitions of $\mathbf{S}, \mathbf{C}$, and $\mathbf{D}$, we can write the payoff for the government as

$$V(a,b,\zeta) = \begin{cases} V_R^+(a,b) & \text{if } (a,b) \in \mathbf{S}, \\ V_R^+(a,b) & \text{if } (a,b) \in \mathbf{C} \;\&\; \zeta \in \{0\}, \\ V_D(a) & \text{if } (a,b) \in \mathbf{C} \;\&\; \zeta \in \{1\}, \\ V_D(a) & \text{if } (a,b) \in \mathbf{D}. \end{cases} \tag{8}$$

When $(a,b) \in \mathbf{C}$, the equilibrium outcome is undetermined, and the government's payoff depends on the sunspot. Notice that the equilibrium payoff for the government never takes the value V_R^-, as this is an off-equilibrium payoff. Nonetheless, V_R^- is essential for determining which zone the government is in and thus the government's ultimate payoff.

We now analyze the equilibrium payoffs in each zone.

Safe zone. In Cole and Kehoe (2000) under $\beta(1+r) = 1$, when the government is in the safe zone, it stays in the safe zone with a constant level of debt and consumption. The logic is that once the government reaches the safe zone, it can achieve the level of consumption that would prevail in the absence of default risk. Crucial for this result is that if the government were to choose a portfolio outside the safe zone, the bond price would fall, reflecting the positive probability of default, and the government would incur the expected costs of defaulting. Moreover, without reserves, postponing default is never optimal: if the government were to leave the safe zone in order to default tomorrow, it would be strictly better off defaulting today.

In our model with reserves, maintaining a constant path of consumption in the safe zone requires that for any initial $(a, b) \in \mathbf{S}$, consumption is given by $c = y + (1-\beta)(a-b)$, and any portfolio in the safe zone satisfying $a' - b' = a - b$ is consistent with this path. Because utility is concave, it follows directly that any other portfolio in the safe zone that induces non-stationary consumption yields strictly lower lifetime utility. With access to initial reserves, however, it is possible that postponing default could be optimal. The government could use reserves to service current coupon payments, postpone the realization of default costs, and thereby deliver a smoother consumption path. This logic connects with Bulow and Rogoff (1989), where access to reserves can reduce debt sustainability.

We henceforth restrict attention to policy strategies in which the government remains in the safe zone once it arrives there, and never transitions from **C** to **D**. In Appendix D, we provide a condition for δ under which this assumption is indeed consistent with optimality.[18]

We then have that the value of being in the safe zone is

$$V(a, b) = \frac{u(y + (1-\beta)(a-b))}{1-\beta} \equiv V_S(a-b), \tag{9}$$

and the policy functions satisfy

$$c(a, b) = y + (1-\beta)(a-b) \equiv c_S(a-b),$$
$$a'(a, b) - b'(a, b) = a - b.$$

with $(a'(a, b), b'(a, b)) \in \mathbf{S}$. Notice that the payoff and the policy for the government depend only on the net foreign asset (NFA) position, defined as $a - b$, and not on the gross positions.

Default zone. Given our assumption that $\beta(1+r) = 1$, under default, the government keeps reserves constant and consumes its income plus the annuity value of the reserves. Thus, the value of default is given by

$$V_D(a) = \frac{u(y + (1-\beta)a) - \phi}{1-\beta}. \tag{10}$$

Crisis zone. Finally, we turn to the equilibrium payoffs in the crisis zone. Recall that the problem the government faces when investors are willing to roll over the bonds is given by (3).

[18]As discussed in Appendix D, for $a = 0$, leaving the safe zone is never optimal under any configuration. For $a > 0$, we show that there is a threshold for δ above which this can be ruled out (see Proposition D.1). Intuitively, a policy strategy where the government lowers reserves, repays today, and defaults tomorrow with probability one becomes less likely the shorter the maturity, since paying today's coupons and then defaulting tomorrow becomes less attractive.

Using the results above, we can write the continuation value for the government as follows:

$$\mathbb{E}V(a', b', \zeta') = \begin{cases} V_S(a' - b') & \text{if } (a', b') \in \mathbf{S}, \\ (1-\lambda)V_R^+(a', b') + \lambda V_D(a') & \text{if } (a', b') \in \mathbf{C}, \\ V_D(a') & \text{if } (a', b') \in \mathbf{D}. \end{cases} \tag{11}$$

That is, if the government chooses a portfolio in the safe zone, the continuation value is given by $V_S(a' - b')$. If the government chooses a portfolio in the crisis zone, then the government will obtain $V_D(a')$ with probability λ and $V_R^+(a', b')$ with probability $1 - \lambda$. Finally, if the government chooses a portfolio in the default zone, its continuation value is given by $V_D(a')$.

Staying in the crisis zone is costly for the government. If the government faces a good sunspot tomorrow, it pays ex post a high interest rate on the debt. If the government faces a bad sunspot tomorrow, it does not pay the debt, but it faces the cost of default. While the government pays an actuarially fair interest rate to investors, it still bears the cost of defaulting. As shown by Cole and Kehoe (2000), the government therefore has incentives to deleverage until it reaches the safe zone.[19] Exiting the crisis zone, however, is also costly. To the extent that the utility function is strictly concave, the government may try to exit the crisis zone slowly. Depending on the realization of the sunspot, the government may eventually be able to reach the safe zone, or it may default along the way. How fast the government attempts to exit depends on the perceived probability of facing the bad sunspot. In particular, a higher λ induces a faster exit of the crisis zone.[20]

Under the operating assumption that the safe zone is an absorbing state, we can iterate on the investors' break-even condition (4) to arrive at an expression for the bond price faced by the government. In particular, we can use the fact that the bond price becomes $1/(1 + r)$ in the period in which the government chooses a portfolio that takes it to the safe zone in the next period. Starting from $t = 0$, suppose the government chooses next period's portfolio (a', b') and that, conditional on no bad sunspot realizations up to $T - 1$ (i.e., $\zeta_t = 0$ for all

[19]As explained in Aguiar and Amador (2013), the key element is that default is not a zero-sum outcome. If a government is indifferent between repaying and defaulting and randomizes between the two, it imposes losses on bondholders without generating any offsetting gain for the government, which is indifferent at the margin.

[20]As in Cole and Kehoe (2000), as the probability λ gets close to zero, a government may decide to avoid any increase in savings necessary to escape the crisis zone, and hence it will eventually default once the bad sunspot is realized. In our model with reserves, there is another possibility: the government may draw down the reserve holdings and move to the default region. In Appendix D, we provide a condition for δ that rules this out in a Markov equilibrium. Under our baseline calibration, this requires debt maturities exceeding 200 years, well above the empirical range.

$t = 0, \ldots, T-1$), the government's policy implies that it reaches the safe zone in period T.[21] Then, for any $T > 0$, the bond price satisfies

$$q(a', b') = \left(\frac{\delta + r}{1+r}\right) \sum_{t=1}^{T-1} \left(\frac{1-\lambda}{1+r}\right)^t (1-\delta)^{t-1} + \left[\frac{(1-\lambda)(1-\delta)}{1+r}\right]^{T-1} \frac{1}{1+r}. \tag{12}$$

The first term captures the bond coupon payments investors expect to receive, and the second term reflects the risk-free price of the bond once the government exits the crisis zone.

In the Cole and Kehoe model with one-period debt, consumption is constant over time while the government tries to exit the crisis zone, and once the government reaches the safe zone, consumption increases and then stays constant thereafter. The idea is that keeping consumption low allows the government to reduce its debt and reach the safe zone—and once the government is safe, debt can be kept constant. Proposition 1 provides a similar result in our economy.

Proposition 1 (Monotonically increasing consumption path)**.** *Consider an initial portfolio* $(a_0, b_0) \in \mathbf{C}$ *such that the government exit time is* T*. Then, if* $\zeta_t = 0$ *for all* $t \leq T-1$*, we have* $c_{t+1} \geq c_t$ *for all* $t \leq T$*.*

Proof. In Appendix A.2. □

A subtle difference with the canonical model is that consumption may be strictly increasing throughout the transition out of the crisis zone. This result is due to the dilution effect from long-term debt. In particular, the bond price at which investors are willing to lend depends on the expected T at which the government will exit (see eq. 12). By keeping current consumption low relative to that of the next period, the government may reduce the exit time T and obtain a higher price for its bond issuances. Hence, the government will choose an upward path for consumption if a small decrease in current consumption (and borrowing) allows the government to reduce the exit time. We will illustrate this in Section 3.4.[22]

2.5 Optimal Portfolio during a Run

We have the following lemma.

[21] As in Aguiar and Amador (2013), we assume that if the government is indifferent between policies that imply different exit periods, it chooses the one with the lower exit period.

[22] In the canonical model, the constant path for consumption within the crisis zone is due to the discreteness of time. In a continuous time model, the exit time varies continuously with the level of debt, and thus consumption is strictly increasing within the crisis zone (see Aguiar and Amador, 2023).

Lemma 2 (Optimal policy in a run)**.** *Consider a government facing a run given an initial portfolio* (a, b). *Then,*

(i) Suppose that $a < \delta b$. *If* $(0, (1-\delta)b) \in \mathbf{S}$, *the optimal policy is* $a' = 0$ *and* $b' = (1-\delta)b$

(ii) Suppose that $a \geq \delta b$. *If* $(a - \delta b, (1-\delta)b) \in \mathbf{S}$, *the optimal policy satisfies* $a' - b' = a - b$, *with* $(a', b') \in \mathbf{S}$, $a' \geq 0$, *and* $b' \leq (1-\delta)b$.

Moreover, if $a \geq \delta b$ *and* $(a - \delta b, (1-\delta)b) \in \mathbf{S}$, *then* $b^-(a) = b^+(a)$. *Otherwise,* $b^-(a) < b^+(a)$.

Proof. In Appendix A.3. □

Result (i) says that when the government starts with few reserves and faces a run, the optimal portfolio will be at a corner with zero reserves and zero debt issuances, if this portfolio keeps the government safe from a run next period. The intuition is that when the government cannot roll over the bonds and starts with low reserves, it chooses the maximum consumption possible if it anticipates that it will be safe in the future. Note, however, that if the government remains vulnerable tomorrow after repaying the coupons during a run, it may choose to further reduce current consumption and increase savings (either by accumulating reserves or buying back debt) in order to reduce its vulnerability in the future.[23] In this case we have $b^-(a) < b^+(a)$. That is, a government under a run defaults at strictly lower levels of debt.

Result (ii) says that if the government has a sufficiently high level of reserves, it can achieve the unconstrained level of consumption even when investors refuse to roll over the debt. Notice that this result requires that the government not be vulnerable tomorrow after repaying the coupons due today. This is because the reduction in reserves needed to make the coupon payments today could make the government vulnerable tomorrow, which would induce it to save more.

One interesting observation from (ii) is that reserves do not need to be large enough to pay for all coupons in order to be safe from a rollover crisis. That is, given $r > 0$, there exists

[23]If a government has an initial portfolio at $(0, b^-(0))$, faces a run, and repays the coupons, it will be safe next period. This is because $V_R^-(0, (1-\delta)b^-(0)) > V_R^-(0, b^-(0)) = V_D(0)$, where the inequality follows from the monotonicity of V_R^- (Lemma B.1) and the equality follows from the definition of b^-. Similarly, as δ approaches one, there is little debt left after repaying today, so the government is safe in the next period. For our baseline calibration, the government is indeed safe in the period following the run. However, none of the theoretical results that follow depend on this particular case. See Appendix G for further discussion and for numerical configurations under which a government remains vulnerable in the next period following a run.

$a \in \left[\delta b, \left(\frac{\delta+r}{1+r}\right) b\right]$ such that the government is safe. To be safe, the government needs just enough reserves to repay the fraction of the debt that would allow it to keep the NFA constant. In effect, when the government has a negative NFA, it uses a fraction of the endowment to pay the interest on the debt and keep the principal constant.[24]

3 Reserve Management under Rollover Crises

In this section, we turn to the main question we investigate in this paper: How should the government choose its portfolio to manage the risk of a rollover crisis?

3.1 The Three Zones

We begin by showing how the initial portfolio determines the zone in which the economy lies. Figure 1 illustrates the results.[25] The blue solid line and the red dashed line denote $b^+(a)$ and $b^-(a)$, respectively. These curves define the boundaries between the crisis and default zones, and between the crisis and safe zones. A government with portfolio $(a, b^-(a))$ is indifferent between default and repayment when it cannot roll over its debt, whereas a government with portfolio $(a, b^+(a))$ is indifferent when rollover is possible. As shown in the figure, $b^+(a) \geq b^-(a)$ and for sufficiently large reserves we have $b^+(a) = b^-(a)$, consistent with Lemma 2. That is, for large a, the crisis zone disappears, and defaults occur only due to fundamentals.

Consider any pair (a, b) at the $b^-(a)$ boundary. Holding a fixed, we can see that a lower level of debt takes the government to the safe zone, and a higher level of debt takes it to the crisis zone. Moreover, we can see that $b^-(a)$ has a positive slope: if we hold b fixed, a higher level of reserves takes the government to the safe zone, while a lower level of reserves puts the government in the crisis zone. The figure shows that $b^+(a)$ also has a positive slope, but it is less steep than $b^-(a)$. To understand better these results, we totally differentiate (6) and (7) and obtain

$$\frac{\partial b^-(a)}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^-\left(a,b^-(a)\right)}{\partial a}}{\frac{\partial V_R^-(a,b^-(a))}{\partial b}} \quad \text{and} \quad \frac{\partial b^+(a)}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^+\left(a,b^+(a)\right)}{\partial a}}{\frac{\partial V_R^+(a,b^+(a))}{\partial b}}. \tag{13}$$

[24]To see this more clearly, suppose the government starts with the portfolio $(\delta b, b)$ and thus the NFA is equal to $-(1-\delta)b$. If the government cannot roll over the debt, we have that if it sets $b' = (1-\delta)b$ and $a' = 0$, it keeps the same NFA and achieves the ideal level of consumption; i.e., $c = y - (1-\beta)(1-\delta)b$.

[25]Throughout, we use calibrated parameter values listed in Table 1, described below.

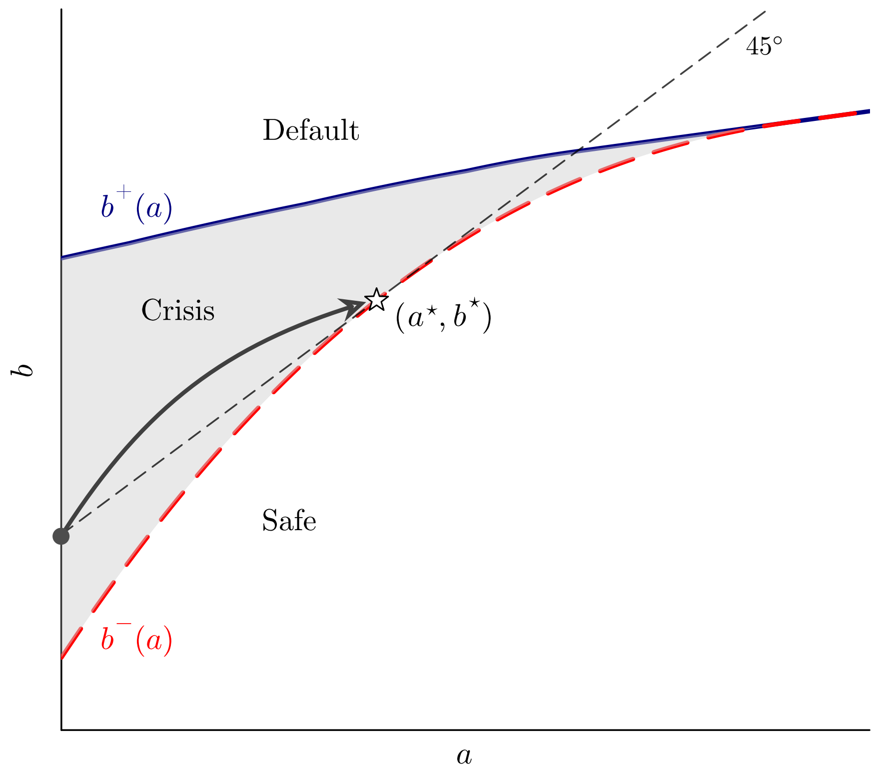

Figure 1: The three zones. The boundaries $b^-(a)$ and $b^+(a)$ are defined in (6) and in (7), respectively.

The slopes of $b^-(a)$ and $b^+(a)$ are determined by how much the value functions of repayment and default change when we vary a and b at the indifference points.[26] As we show formally below, the signs of the two slopes must be positive. Since both the value of repayment and the value of default are strictly increasing in reserves, and the value of repayment is strictly decreasing in debt (see Auxiliary Lemma B.1), a positive slope requires that an increase in reserves raises the value of repayment *by more than* the value of default at the indifference points. The key observation is that, at the b^- and the b^+ thresholds, current consumption under repayment is strictly lower than under default. A marginal increase in reserves can therefore be allocated to current consumption under repayment, raising its value by the marginal utility at a relatively low level of consumption. Under default, the same increase in reserves raises consumption more gradually over time and is valued at a lower marginal utility. By the strict concavity of the utility function, the marginal effect of reserves on the value of repayment strictly dominates, implying that the debt thresholds are increasing in reserves.

[26] The value functions of repayment are not necessarily everywhere differentiable. The formal proof presented below, however, does not rely on differentiability.

We summarize these results in the following proposition.

Proposition 2 (Monotonicity)**.** *In any Markov equilibrium, $b^{-}(\cdot)$ and $b^{+}(\cdot)$ are strictly increasing in a, for all a.*

Proof. In Appendix A.4. □

We have seen that reserves help expand the safe zone. Even though this suggests that reserves could be desirable, in principle, the government could also reduce its vulnerability by decreasing debt instead. The key issue is whether increasing both debt and reserves leads to an increase in $V_R^{-}(a,b)$ that offsets the increase in $V_D(a)$. More concretely, consider a government portfolio that lies slightly above the red, dashed curve in Figure 1. Does an increase in debt and reserves by one unit push the government into the crisis zone or the safe zone? We examine this question next.

3.2 A Joint Increase in Debt and Reserves

Each of the portfolios in the $b^{-}(a)$ boundary is associated with a different NFA. It will be useful to examine the portfolio in the safe zone with the *lowest* NFA position. We refer to this portfolio as the "lowest-NFA safe portfolio," and we denote it by $(a^{\star}, b^{\star})$.

Definition 2. The lowest-NFA safe portfolio $(a^{\star}, b^{\star})$ is the portfolio in the safe zone with the lowest net foreign asset position.

Formally, the lowest-NFA safe portfolio is given by

$$(a^{\star}, b^{\star}) = \underset{a\geq 0, b}{\operatorname{argmin}} \; a - b \qquad \text{s.t.} \quad (a,b) \in \mathbf{S}. \tag{14}$$

Using that $(a,b)\in\mathbf{S}$ if $b \leq b^{-}(a)$ and assuming a *strictly interior solution for $a^{\star}$*, we obtain that the lowest-NFA safe portfolio satisfies $\frac{\partial b^{-}(a^{\star})}{\partial a} = 1$. This portfolio corresponds to the point highlighted in Figure 1 where the 45° line is tangent to the red, dashed line. It is then immediate that if the government starts from a point on or below this tangent line, it can move to the safe zone by increasing reserves and debt by the same amount (thus keeping the same NFA). The arrow in the figure illustrates how a government at the tangent line with zero initial reserves can jump to the safe zone by choosing $(a^{\star}, b^{\star})$.

The lowest-NFA safe portfolio will constitute a focal point. When a government is deep in the crisis zone (i.e., it is above the aforementioned tangent line), it needs to increase its NFA to reach the safe zone. The larger the required increase in the NFA, the higher the cost of exiting the crisis zone, because of the concavity of the utility function. The portfolio $(a^\star, b^\star)$ makes the government safe and minimizes the need for consumption cuts.

The key question we tackle next is, when do we have $a^\star > 0$?

A condition for $a^\star > 0$. The proposition below provides a sufficient condition guaranteeing that the solution to (14) is strictly interior.

Proposition 3 (Positive reserves)**.** *Suppose that the boundary of the crisis zone at zero reserves, $b^-(0)$, satisfies the following condition:*

$$\beta(1-\delta)\left[u'\left(y - \left(\frac{\delta + r}{1+r}\right) b^-(0)\right) - u'\left(y - (1-\beta)(1-\delta)b^-(0)\right)\right] > u'(y). \qquad (15)$$

Then, the lowest-NFA safe portfolio features a strictly positive level of reserves. That is, $a^\star > 0$.

Proof. We first show that from a starting point of zero reserves, an increase in reserves raises the b^- threshold by more than one unit,

$$\left.\frac{d\, b^-(a)}{da}\right|_{a=0} > 1\,, \qquad (16)$$

where note that the derivative exists by the implicit function theorem and the fact that both V_D and V_R^- are continuously differentiable in a neighborhood around $a = 0$.

To prove (16), we start by noting that taking the envelope condition in (5) and evaluating it at $(0, b^-(0))$, we obtain

$$\frac{\partial V_R^-\left(0, b^-(0)\right)}{\partial b} = -\, u'\left(y - \left(\frac{\delta + r}{1+r}\right) b^-(0)\right)\left(\frac{\delta + r}{1+r}\right) - \beta(1-\delta)u'\left(y - (1-\beta)(1-\delta)b^-(0)\right),$$

where the expression uses that $a' = 0$ and that the government is safe the period after the run, both of which follow from part (i) of Lemma 2 and $a = 0 < \delta b^-(0)$.

Replacing the previous equation and the value for $\partial V_D(a)/\partial a$ into the first expression in (13), and using (15), we obtain the inequality in (16).

To prove that $a^\star > 0$ we work toward a contradiction. Suppose that the lowest-NFA safe portfolio $(a^\star, b^\star)$ is such that $a^\star = 0$. Consider an alternative portfolio $(\tilde{a}, b^-(\tilde{a}))$, for

$\tilde{a} > 0$ approximately close to zero. We have

$$b^{-}(\tilde{a}) - b^{-}(0) = \int_0^{\tilde{a}} \frac{\partial b^{-}(a)}{\partial a} da > \int_0^{\tilde{a}} da = \tilde{a}, \tag{17}$$

where the strict inequality follows from (16). Rearranging (17), we arrive at

$$\tilde{a} - b^{-}(\tilde{a}) < 0 - b^{-}(0) .$$

That is, there is a portfolio $(\tilde{a}, b^{-}(\tilde{a})) \in \mathbf{S}$ with an even lower NFA than that of the portfolio with zero reserves $(0, b^{-}(0))$. We thus find a contradiction that $a^{\star} = 0$. That is, the lowest-NFA safe portfolio features strictly positive reserves.

□

The claim is that when condition (15) holds, the slope of $b^{-}(a)$ is locally steeper than the 45-degree line at zero reserves, i.e., $\left.\frac{\partial b^{-}(a)}{\partial a}\right|_{a=0} > 1$. At $a = 0$, a marginal increase in reserves raises both V_R^{-} and V_D, but it raises V_R^{-} by more. The reason is that larger gross positions—holding net foreign assets constant—relax the government's budget constraint during a run, because only a fraction of the debt comes due each period. For a given net foreign asset position, a joint increase in debt and reserves therefore increases the resources available to the government and raises the value of repayment.

At low levels of reserves, concavity implies that the marginal benefit of additional reserves under repayment dominates the combined effect of a higher default value and a higher repayment burden associated with more debt. Formally, for low reserves,

$$\frac{\partial V_R^{-}}{\partial a} > \frac{\partial V_D}{\partial a} - \frac{\partial V_R^{-}}{\partial b} .$$

As reserves increase, diminishing marginal utility reduces the marginal value of additional liquidity in a run, causing the slope of $b^{-}(a)$ to decline. At the point where $\frac{\partial V_R^{-}}{\partial a} = \frac{\partial V_D}{\partial a} - \frac{\partial V_R^{-}}{\partial b}$, we have $\partial b^{-}(a)/\partial a = 1$, which corresponds to the portfolio $(a^{\star}, b^{\star})$. Starting from this portfolio, an equal increase or decrease in reserves and debt—i.e., scaling gross positions up or down while holding net foreign assets fixed—moves the government into the crisis zone.

Maturity also plays a critical role for condition (15) to hold. In particular, the condition can be satisfied only for intermediate values of maturity. In the case of $\delta = 0$, the bond becomes a perpetuity, in which case rollover risk becomes irrelevant as the government pays only interest and never pays any principal. Conversely, in the case of $\delta = 1$, condition (15) cannot be satisfied either. Intuitively, when debt is one-period, the value of repayment for the government depends only on $a - b$ (i.e., it is independent of the gross positions). Because V_D increases with a, it thus follows that a one-unit increase in a and b must lower $V_R^-(a, b) - V_D(a)$. This implies that if a government is indifferent between repaying while facing a run and defaulting, an increase in debt and reserves will always push the economy into the crisis zone. The proposition below formalizes the role of maturity.

Proposition 4. *Suppose that $\delta = 0$ or $\delta = 1$. Then, the lowest-NFA safe portfolio features zero reserves, $a^\star = 0$.*

Proof. In Appendix A.5. □

Condition (15) depends on $b^-(0)$, which is an endogenous object. The next proposition establishes that there exists an intermediate range of δ for which condition (15) holds, thus guaranteeing that reserves are strictly positive.

Proposition 5. *For any $\{\beta, y\}$, there exists $\delta \in (0, 1)$ and $\phi > 0$ such that condition* (15) *holds; therefore $a^\star > 0$.*

Proof. In Appendix A.6 □

The proof of the proposition draws on the result that for any candidate value for the $b^-(0)$, there is a unique default cost ϕ that rationalizes such a debt level and that, in turn, ϕ does not affect condition (15). When the level of $b^-(0)$ is sufficiently large, an Inada condition guarantees that the marginal value of having one more unit of reserves and paying a fraction $\left(\frac{\delta+r}{1+r}\right)$ of the existing stock of debt becomes arbitrarily large. As a result, having one more unit of debt and reserves increases the value V_R^- by more than V_D. This implies that $\left.\frac{\partial b^-(a)}{\partial a}\right|_{a=0} > 1$ holds, which guarantees $a^* > 0$.

We highlight that the proposition is valid for any utility function satisfying the usual properties highlighted above. In Appendix C, we provide an example with quadratic utility, which allows for a closed-form expression for $b^-(0)$ and a simple direct characterization of the set of parameters that guarantee that $a^\star > 0$.

3.3 Exiting the Crisis Zone

We now analyze the optimal strategy for a government attempting to exit the crisis zone. Should it reduce its debt or instead accumulate reserves? And if reserve accumulation is optimal, should reserves be built up gradually or only at the moment of exit?

Iso-T regions. We first examine how long it takes for the government to exit the crisis zone, depending on the initial portfolio. Following the terminology of Aguiar and Amador (2013), we construct "iso-T regions." Starting from $t = 0$, we compute in which period the government will reach the safe zone, conditional on no bad sunspot realizations prior to exit.

Figure 2 shows the exit time T as a function of the initial portfolios (a, b). As we can see, when the gross positions are close to $(a^\star, b^\star)$, the government exits in one period. That is, $T = 1$. As we move up and toward the left (that is, increasing debt and lowering reserves), it takes more periods to exit. Under our benchmark parametrization (see Table 1), we can see regions with exit times $T = 2$, $T = 3$, and $T = 4$. For initial debt levels above $b^+(a)$, the government strictly prefers to default in equilibrium.

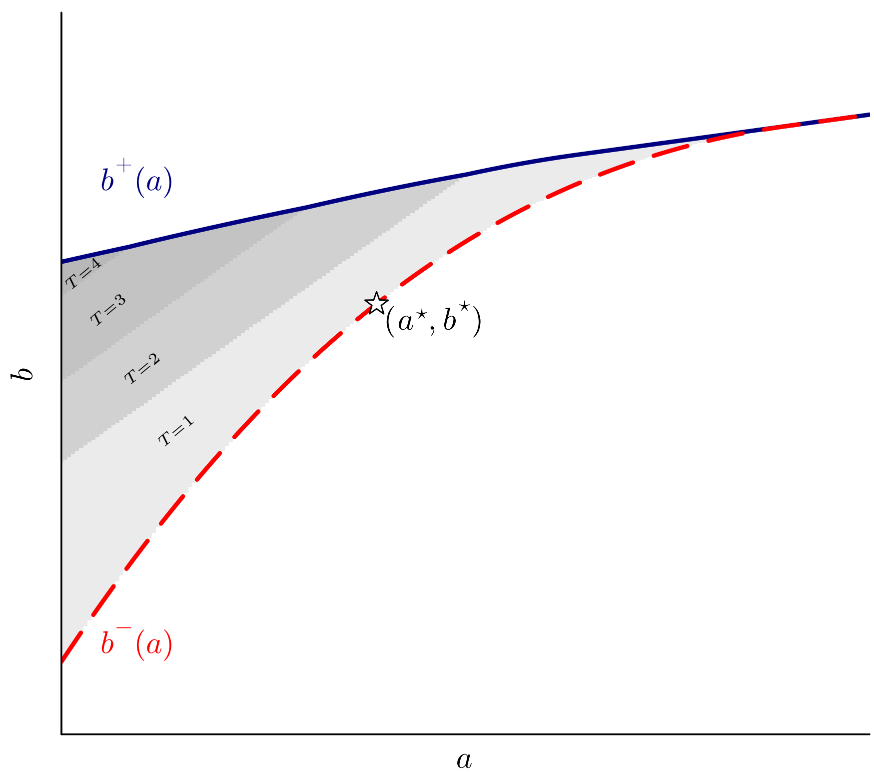

Figure 2: Iso-T regions. The shaded areas describe how many periods it takes for the government to exit the crisis zone.

Optimal portfolio. We now analyze the optimal portfolio choice for the government that is today in the crisis zone and is able to borrow.

The proposition below characterizes the solution.

Proposition 6 (Optimal portfolio)**.** *Consider an initial* $(a, b) \in \mathbf{C}$*. The optimal portfolio choice* $\{a'(a, b), b'(a, b)\}$ *that solves* (3) *satisfies the following conditions:*

(i) Suppose that $a - b < a^\star - b^\star$*. Then, if* $(a'(a, b), b'(a, b)) \in \mathbf{S}$*, we have that* $a'(a, b) = a^\star$ *and* $b'(a, b) = b^\star$*.*

(ii) Suppose that $(a'(a, b), b'(a, b)) \in \mathbf{C}$*. Then, we have that* $a'(a, b) = 0$*.*

Proof. In Appendix A.7. □

Part (i) considers a government in the crisis zone that has an NFA that is lower than the lowest-NFA safe portfolio. The proposition shows that if the government chooses a portfolio that puts it in the safe zone, then the government chooses the lowest-NFA safe portfolio $(a^\star, b^\star)$. The idea is that choosing $(a^\star, b^\star)$ allows the government to exit the crisis zone, minimizing the cut in consumption. Portfolios in the safe zone with lower amounts of reserves would imply the government needs to reduce debt by more than the reduction in reserves. Portfolios in the safe zone with higher amounts of reserves would allow the government to borrow more, but the increase in borrowing would be smaller than the increase in reserves.

Part (ii) considers a situation where the government chooses a portfolio that keeps it in the crisis zone in the next period. The proposition shows that it is not optimal to accumulate reserves in this case. To establish the result, we show that any policy strategy with positive reserves can be improved by an alternative policy. In particular, we consider an alternative strategy where the government issues less debt and accumulates lower amounts of reserves, in such a way that keeps the *continuation value under repayment unchanged.*[27] The exact changes in the portfolio that deliver this goal lead to an increase in current consumption, but, with lower future reserves, also lead to a decrease in consumption under an eventual default next period. Because the government is trying to deleverage toward the safe zone, consumption under repayment is strictly lower than consumption under default; hence, the alternative strategy improves welfare over the initial policy with $a' > 0$.[28] Therefore, issuing debt to accumulate reserves in the crisis zone is strictly suboptimal as long as the government remains in the crisis zone.

[27] This perturbation is in the spirit of the zero-cost trades of Aguiar and Amador (2013), but here we keep the continuation value under repayment constant rather than consumption. Their model features local indeterminacy of short- and long-term portfolios to the extent that the bond price does not change, but this is not the case in our environment. In Aguiar and Amador (2025), they show how a swap between short-term and long-term debt can help a government escape the crisis zone.

[28] A reader may wonder whether this result relies on the assumption that the cost of default is in terms of utility and not resources. We can show, however, that the same result holds under an income cost of defaulting.

Putting these two parts together establishes that *the optimal exit strategy is to delay the accumulation of reserves until the government is ready to exit the crisis zone.*[29] We summarize this in the following corollary.

Corollary 1 (Optimal exit strategy)**.** *Consider a portfolio* $(a, b) \in \mathbf{C}$ *such that* $a - b < a^\star - b^\star$ *and the government exits after* T *periods for* $T < \infty$*, provided that* $\{\zeta_t\}_{t=0}^{T-1} = 0$*. Then, we have* $a_{t+1} = 0$ *for all* $t < T - 1$ *and* $a_T = a^\star$*,* $b_T = b^\star$*.*

Proof. In Appendix A.8. □

We turn next to the case when the government has an initial NFA higher than the one in the lowest-NFA safe portfolio. The proposition below shows that the government exits the crisis zone in one period.

Proposition 7 (Immediate exit)**.** *Consider an initial portfolio* $(a, b) \in \mathbf{C}$ *and suppose that* $a - b \geq a^\star - b^\star$*. Then, exiting in one period achieves the optimal allocation. Moreover, any portfolio* $(a', b') \in \mathbf{S}$ *such that* $a - b = a' - b'$ *is optimal.*

Proof. In Appendix A.9. □

The logic for why the government exits in one period is that it is feasible for the government to choose a portfolio that takes it to the safe zone without having to reduce consumption relative to the unconstrained optimal. That is, exiting the crisis zone is not costly in this situation. Notice that in this case, there is a range of portfolios that are optimal, including $(a^\star, b + a^\star - a)$ in particular.

Illustration. Figure 3, presents three simulations: one in which the government exits the crisis zone in one period (panel [a]), one in two periods (panel [b]), and one in three periods (panel [c]). Consistent with Proposition 6, we can see that when the government exits in two periods, it chooses zero reserves in period 1, and when the government exits in three periods, it chooses zero reserves in periods 1 and 2.

[29] Note that under continuous time, we would still obtain upon exit a discrete jump in reserves to the extent that it is financed with a discrete increase in debt.

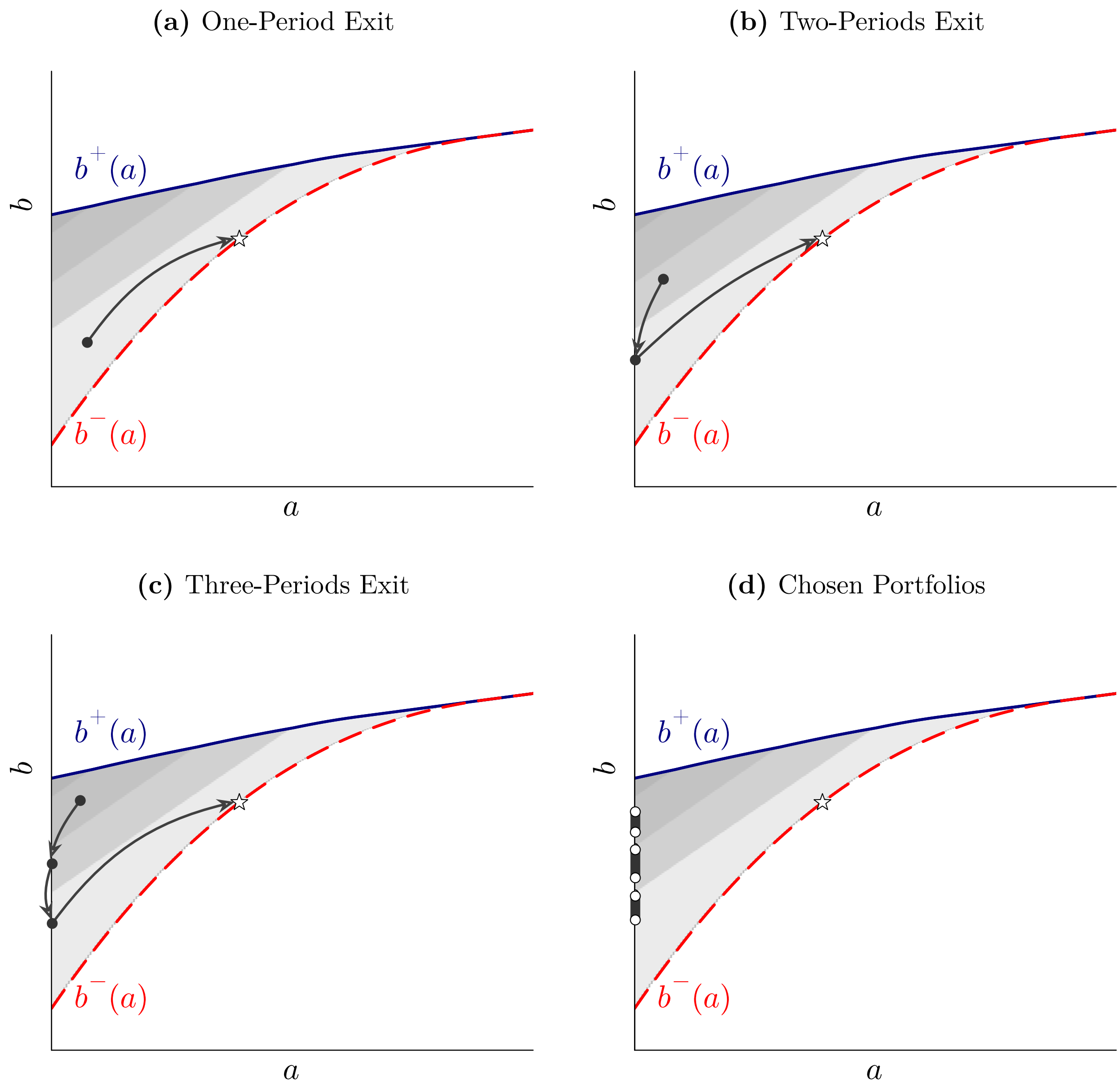

Figure 3: Periods to exit and portfolios. The figure presents examples where the government exits in one period (panel a), two periods (panel b), and three periods (panel c). The solid dots in panel (d) represent all the possible portfolios the government chooses for any initial portfolio in the crisis zone such that $a - b < a^\star - b^\star$. The star in all four panels indicates the portfolio $(a^\star, b^\star)$.

Panel (d) of Figure 3 displays all portfolios chosen along the transition for the full set of initial portfolios with NFA lower than that of the lowest-NFA safe portfolio. As the panel illustrates, the government never accumulates reserves unless doing so places it directly in the safe zone. The figure also exhibits "holes," in the sense that a range of b' values are never chosen in equilibrium. For these values, the government prefers either to borrow slightly less and exit sooner (thereby obtaining a higher bond price) or to borrow slightly more and exit in the same number of periods, which improves intertemporal consumption smoothing.

Policy correspondences. To further inspect the portfolio of the government, Figure 4 presents the policies for reserves, borrowing, NFA, and consumption in panels (a), (b), (c), and (d), respectively. These policies correspond to the solution of the government problem (3) (i.e., they are the policies conditional on repayment and having access to the bond market). The plots are presented for a range of initial values for b and for $a = 0$. The gray areas denote correspondences.

The figure highlights three vertical lines: the first vertical line denotes $b^{-}(0)$, the middle vertical line denotes $b^{-}(a^{\star}) - a^{\star}$, and the third vertical line denotes $b^{+}(0)$. The first vertical line indicates the boundary between the safe zone and the crisis zone, for zero initial reserves. For debt levels to the left of this line, the government is already in the safe zone, and there is a range of portfolios that deliver the optimal solution, as illustrated in the shaded area in panels (a) and (b). Consumption and the NFA, however, are uniquely determined (panels [c] and [d]). In particular, consumption is given by $c = y - (1 - \beta)b$, and the NFA equals $-b$.

The middle vertical line corresponds to a portfolio with the same NFA as the lowest-NFA safe portfolio. For debt levels between the first and the middle vertical line, the government is initially in the crisis zone. Because the government starts from an NFA position that is higher than the lowest-NFA safe portfolio, it can exit the crisis zone immediately and without cutting consumption, as implied by Proposition 7. As in the previous case, optimal portfolios are correspondences, and the NFA is kept constant. However, in this region, *the government must accumulate a strictly positive amount of reserves to exit the crisis zone*. Moreover, it finances reserve accumulation through new debt issuances.

For debt levels immediately to the right of the middle vertical line, the government still exits in one period, but it now needs to increase the net foreign asset position to reach the safe zone. As is consistent with Proposition 6, the government finds it strictly optimal to choose $(a^{\star}, b^{\star})$. One can also see that as the initial debt level increases further, there is a sharper drop in consumption (panel [d]). This is intuitive because while the initial debt level is increased, the government continues to choose the portfolio $(a^{\star}, b^{\star})$. It is also interesting to note that the policy function for debt lies above the 45-degree line (panel [b]). That is, *the government increases the amount of debt to exit the crisis zone.*

As we move to the right (still between the middle and the third vertical lines), borrowing falls discretely, and the government chooses zero reserves. At this point, the government now takes two periods to exit. We can also see that now the policy for b' lies below the 45° line. That is, the government reduces its debt and delays the accumulation of reserves. As

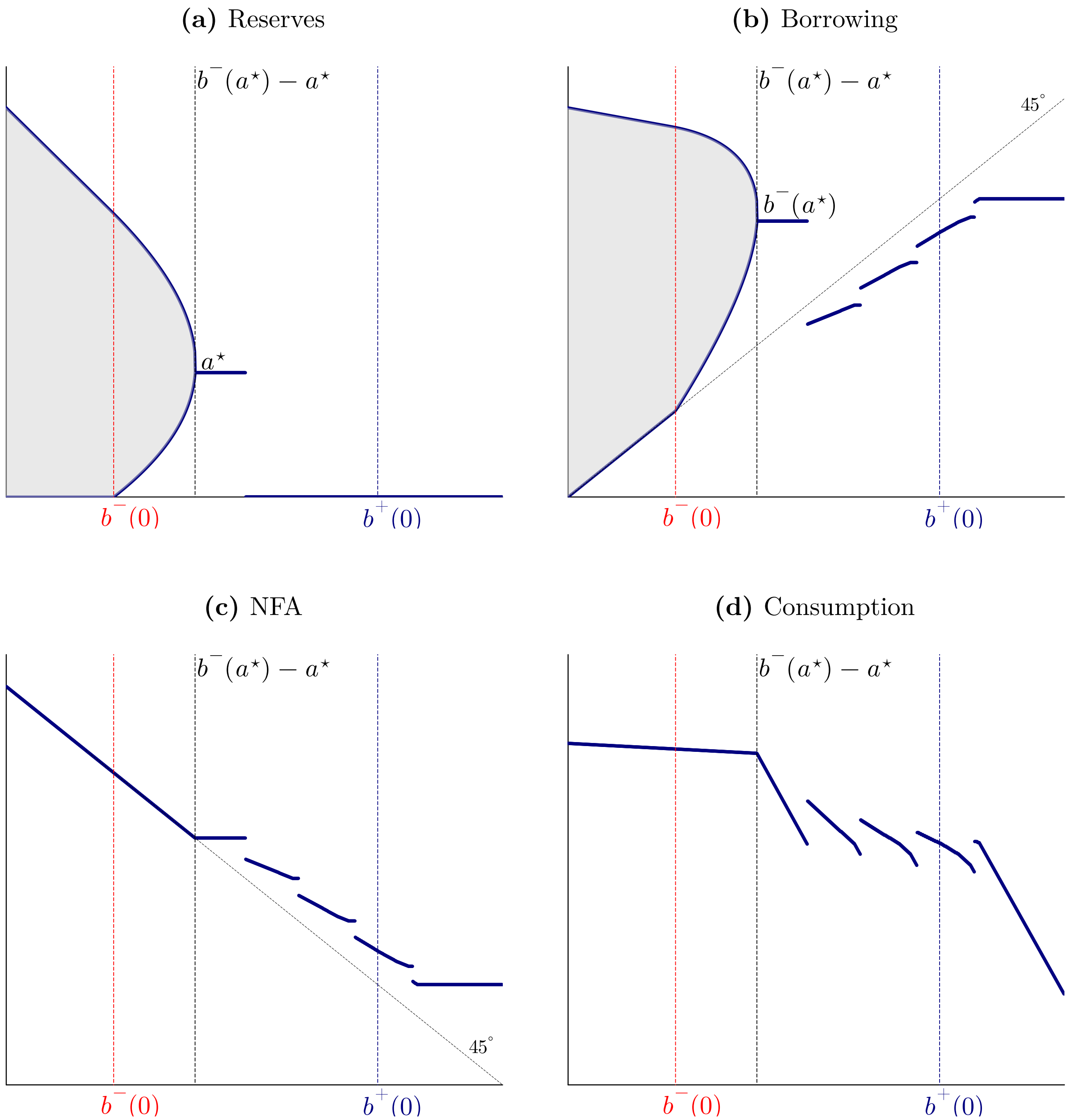

Figure 4: Policy correspondences. The plots are for an initial level of reserves $a = 0$ and a range of values for initial debt (the horizontal axis). The top panels denote $a'(0, b)$ and $b'(0, b)$, and the bottom panels denote $a'(0, b) - b'(0, b)$, and $c(0, b)$. Light gray areas represent the portfolio indeterminacy in that region.

we move further to the right, we see another point of discontinuity when the government postpones by one more period the planned exit of the crisis zone. Specifically, at the point of discontinuity, the government is indifferent between choosing a certain level of borrowing and a significantly higher one that delays the exit time by one period. When the government chooses to postpone the exit by one period, we can see that consumption jumps upward as we increase the initial debt (and then falls continuously with the initial debt level).[30] Once debt exceeds the third vertical line, denoting $b^{+}(0)$, the government chooses to default in equilibrium, but recall that the figure presents the policy conditional on repayment. Notice also that in the figure, the choice of borrowing never exceeds $b^{+}(0)$ (because the government would face a zero bond price in that case).

To conclude, we have characterized the optimal strategy for a government that seeks to exit the crisis zone. The key takeaway is that when the government is deep in the crisis zone, the optimal policy is to reduce the debt and keep zero reserves. As the government approaches the safe zone, it is optimal to increase borrowing and accumulate reserves.

3.4 Quantitative Results

In this section, we present a calibration of the model to assess the quantitative role of reserves. We use data from Italy to calibrate the model. Parameter values are presented in Table 1.[31]

A model period is one year, and income is normalized to one. As in Conesa and Kehoe (2017) and Bocola and Dovis (2019), we assume the utility function takes the form

$$u(c) = \frac{(c - \underline{c})^{1-\sigma}}{1-\sigma},$$

where $\underline{c}$ stands for a minimum level of consumption that cannot be changed in the short run. Following Bocola and Dovis (2019), we set $\underline{c} = 0.68$. This value is based on the measure of non-discretionary spending for the Italian government. We set $\sigma = 2$ and $r = 0.03$, which are common values in the literature.

We calibrate the cost of default, ϕ, so that the midpoint between $b^{-}(0)$ and $b^{+}(0)$ is roughly 100% of GDP, the level of Italy's debt in the run-up to the sovereign debt crisis

[30]The saw-tooth pattern for NFA and consumption in panels (c) and (d), respectively, shares some common features with Cole and Kehoe's model with one-period debt and no assets. One subtle difference with long-term debt is that the policy function for NFA has several flat regions. This is related to the debt dilution effect of long-term debt, which implies that the government is deterred from a slight increase in borrowing despite the higher initial stock of debt. This is because higher borrowing would increase the exit time and discretely reduce the bond price. Notice that at these flat portions, consumption experiences a sharper drop.

[31]The numerical algorithm is described in Appendix H.

Table 1: Parameter values

Parameter	Value	Description	Source
y	1	Endowment	Normalization
σ	2	Risk-aversion	Standard
r	3%	Risk-free rate	Standard
$1/\delta$	6	Debt maturity	Italian Debt
β	0.97	Discount factor	$\beta(1+r)=1$
$\underline{c}$	0.68	Consumption floor	Bocola and Dovis (2019)
λ	0.5%	Sunspot probability	Baseline
ϕ	0.33	Default Cost	Debt-to-income = 100%

in 2012.[32] The maturity of the debt is set to 6 years, which implies $\delta = 1/6$. Finally, we use $\lambda = 0.5\%$ as a baseline value. This value is important for the speed at which the economy exits the crisis zone but does not affect the lowest-NFA safe portfolio $(a^\star, b^\star)$ for this parametrization.[33]

Simulation results. We obtain that the lowest-NFA safe portfolio is $a^\star = 0.08, b^\star = 1.02$. Recall that we normalize income to one, so the values can be interpreted as fractions of GDP. Figure 5 shows the time series simulation for a government that starts in the crisis zone. For the initial portfolio at $t = 0$ considered, the government reaches the safe zone in four periods. For $t \geq 4$, all variables therefore remain constant.

As can be observed in panels (a) and (e), the NFA increases monotonically while the debt level falls initially and then increases at $t = 3$ upon exiting the crisis zone. In addition, consumption (panel [c]) is weakly increasing over time, in line with Proposition 1. In particular, consumption is constant for $t = 0, 1, 2, 3$ and increases at $t = 4$ once the economy reaches the safe zone. Finally, panel (f) illustrates how the bond price increases monotonically over time. As highlighted in the recursion (12), as the government approaches the safe zone, the probability of a future default falls and the bond price increases.

[32]The calibrated default penalty, $\phi = 0.33$, implies a lifetime utility loss equivalent to a permanent 2.65% reduction in consumption starting from the $(a^\star, b^\star)$ portfolio.

[33]See Appendix F for alternative calibrations of λ .

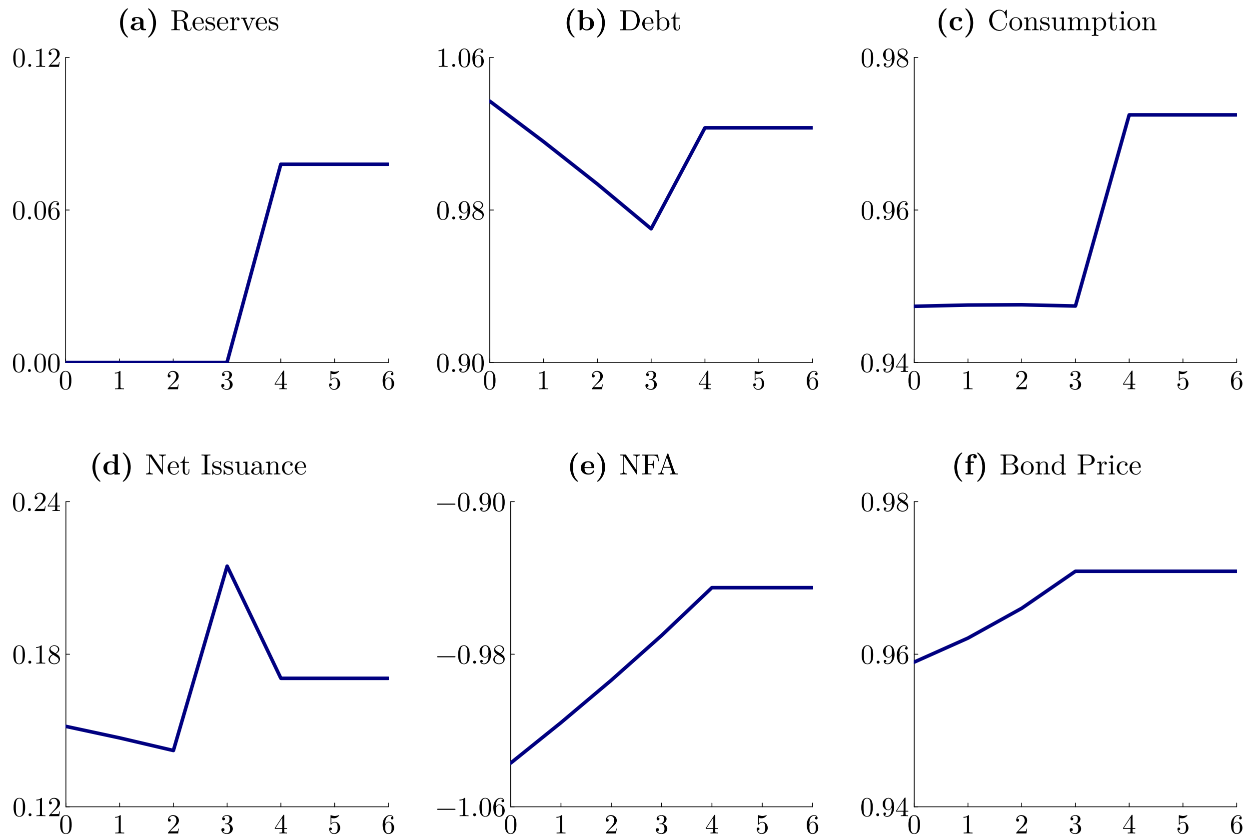

Figure 5: Deleveraging dynamics. The government is assumed to start in the crisis zone with $b = 1.04$ and zero initial reserves. Panels (a), (b), and (e) plot beginning-of-period levels of reserves, debt, and NFA, respectively.

Sensitivity. In Figure 6, we examine how changes in the parameters for the debt duration and risk aversion alter the optimal portfolio for the government. Specifically, we vary δ and σ and recalibrate ϕ to match the same $b^{-}(0)$ as in the baseline calibration. The figure shows the effect that varying these parameters (one at a time) has on the value of $a^{\star}$.

Panel (a) shows that when the debt maturity becomes shorter, the government accumulates a larger amount of reserves. A shorter maturity implies that a larger fraction of the debt becomes due each period.[34] Therefore, the government needs a higher amount of reserves to be safe from a rollover crisis. For a maturity of 4 years, the stock of reserves can reach close to 30% of GDP. Panel (b) shows that for higher degrees of risk aversion, the government accumulates a larger amount of reserves. Higher risk aversion implies greater curvature in the utility function. Thus, reserves provide a higher marginal value when the government faces a run.

[34]Recall, however, that in the limit when the maturity is one period, there is no scope for reserve accumulation, as established in Proposition 4.

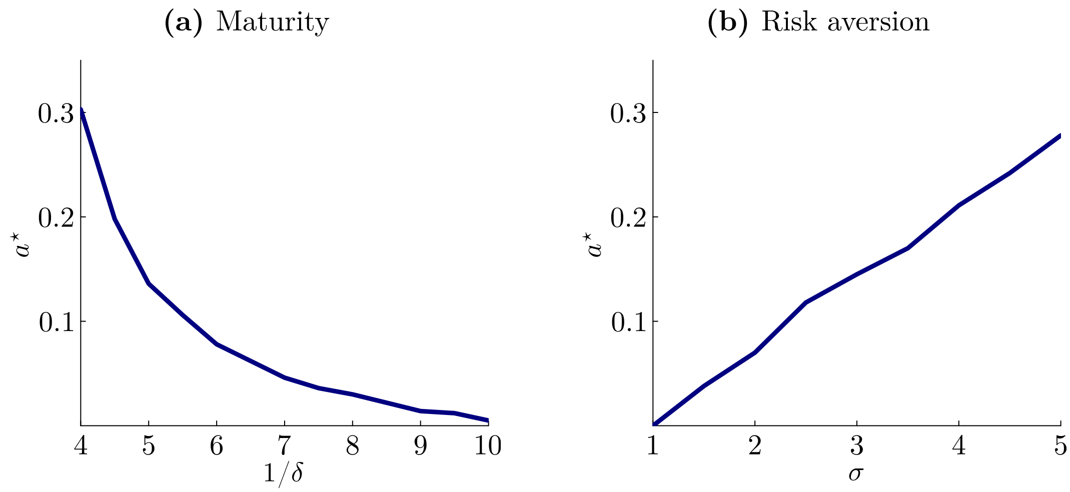

Figure 6: Sensitivity analysis. The panels show the level of reserves $a^\star$ for different parameter values for the (inverse of the) maturity δ and the risk aversion σ. In the simulations, we recalibrate the value of ϕ to match the same debt level $b^-(0)$ as in the baseline calibration.

Counterfactual without reserves. We next compare our benchmark economy with one in which the government never accumulates reserves under repayment. Namely, the government starts with portfolio (a, b) at some t, and from $t+1$ onward, it is forced to set $a' = 0$.[35]

Figure 7 compares the bond price schedules with and without reserves. One can see that the bond price with reserves is always above the economy without reserves, strictly so for a range of values of debt. In the absence of reserves, exiting the crisis zone becomes more costly, as now the government's terminal point is $(0, b^-(0))$ instead of $(a^\star, b^\star)$. That is, the government must either cut consumption more during the deleveraging process or take more time to reach the safe zone. Because exiting is more costly, it follows that the debt threshold b^+ is reduced. The lack of access to reserves makes debt less sustainable.[36]

[35]We impose the restriction to never accumulate reserves only under repayment, so this means that V_D remains the same as in the baseline.

[36]Appendix E presents more details. Figure E.1 presents the policy function for NFA and consumption without reserves, using the baseline parametrization. Figure E.2 presents a simulation where the government without reserves takes more time to exit the crisis zone and chooses lower consumption during this transition as it deleverages.

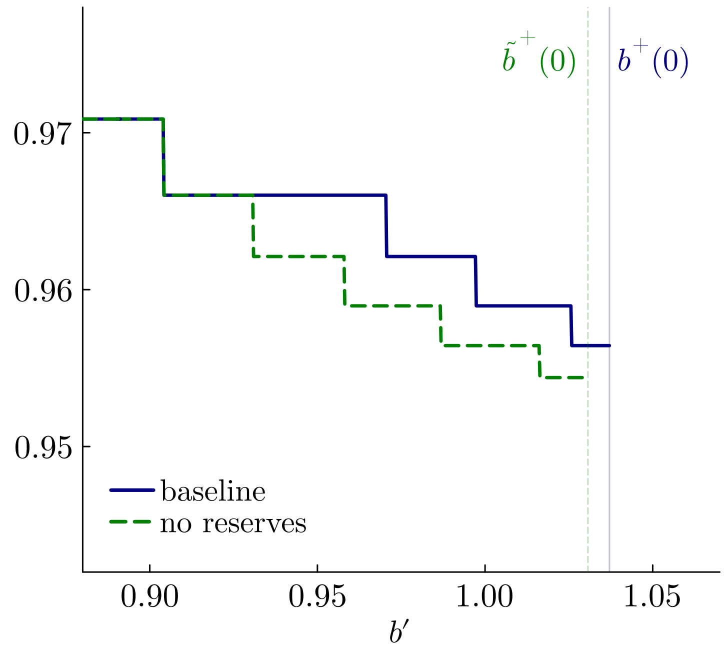

Figure 7: Bond price schedules with and without reserves. The blue solid line denotes the bond price schedule in the baseline model at zero reserves, $q(0, b')$. The green, dashed line denotes the bond price schedule in the economy without reserves, $q(b')$. The green, dashed vertical line $\tilde{b}^+(0)$ is given by $\tilde{V}_R^+(\tilde{b}^+(0)) = V_D(0)$, where $\tilde{V}_R^+$ corresponds to the value function when the government cannot accumulate reserves under repayment in the future.

3.5 Empirical Analysis

In this section, we provide empirical evidence consistent with the model's implications for reserve accumulation. The theory yields two predictions: (i) issuing debt and accumulating reserves can lower sovereign spreads, and (ii) once accumulated, reserves are not drawn down. In the model, the first prediction arises because reserves provide a liquidity backstop that reduces default risk, while the second reflects that reserves are held primarily to deter runs on the equilibrium path—not as insurance to be deployed ex post.

These predictions of our model contrast with studies that point out an insurance role of reserves—in particular Bianchi et al. (2018), where higher gross positions lead to higher spreads and the government depletes reserves in times of fiscal stress. In that framework, higher gross positions provide insurance against increases in borrowing costs. This is because higher reserves allow the government to smooth consumption when a persistent adverse income shock leads to an increase in the probability of future default and sovereign spreads.[37] Whereas in Bianchi et al. (2018) higher gross positions hedge against increases in interest rates, in our model higher gross positions reduce rollover risk and prevent interest rates from rising in the first place.

To assess whether the model's predictions are borne out in the data, we use data on international reserve holdings, external public debt, sovereign spreads, and other macroeconomic variables for a panel of 37 emerging economies. Appendix I provides full details on the dataset. We examine the two predictions in turn.

(i) Issuing debt to accumulate reserves reduces spreads. We test this prediction by estimating a standard spread regression for our panel of countries:

$$\text{Spread}_{it} = \beta_{\text{Res}}\,\text{Reserves}_{it} + \beta_{\text{Debt}}\,\text{Sov.Debt}_{it} + \beta_{\text{Int}}\,\text{Reserves}_{it} \times \text{Sov.Debt}_{it} + \beta'_{\text{M}}\,\text{M}_{it} + \epsilon_{it}\,,$$

where β_{Res}, β_{Debt}, and β_{Int} are the coefficients of main interest, M_{it} is a vector of macroeconomic control variables, and ϵ_{it} is a random error.[38]

Table 2 presents the results. A central finding is that a *joint* increase in sovereign debt and reserves is associated with a reduction in sovereign spreads. Moreover, this effect is concentrated in economies with relatively low debt levels. As shown in columns 3 and 4, the discrete marginal effect of simultaneously increasing debt and reserves by 1 percent of GDP is approximately -10 basis points and statistically significant for the low-debt subsample, while the corresponding effect for the high-debt subsample is smaller and statistically insignificant.[39]

This pattern is consistent with the predictions of the theory: joint increases in debt and reserves can lower spreads to the extent that they reduce the vulnerability to a self-fulfilling crisis.

[37] In particular, states with high spreads are associated with high marginal utility of consumption, and therefore by issuing debt and reserves, the government effectively transfers resources from periods of low marginal utility to periods of high marginal utility.

[38] Following Levy Yeyati (2008) and Sosa-Padilla and Sturzenegger (2023), the vector M_{it} includes measures for global risk aversion, the world interest rate, and private external debt to GDP. We also include country fixed effects and year dummies. Table I.1 in Appendix I reports the full set of coefficients.

[39] We estimate the discrete marginal effect of simultaneously increasing reserves and debt by 1% of GDP as

$$\mathbb{E}[\text{Spread} \mid a_0 + 1, b_0 + 1] - \mathbb{E}[\text{Spread} \mid a_0, b_0] = \hat{\beta}_{\text{Res}} + \hat{\beta}_{\text{Debt}} + \hat{\beta}_{\text{Int}}\big[(a_0 + 1)(b_0 + 1) - a_0 b_0\big],$$

where a_0 and b_0 denote the reference values of reserves and debt (in percent of GDP), respectively. We evaluate the marginal effects at $a_0 = 0$ and b_0 equal to the mean value in the data. The estimated marginal effects are similar when evaluated at the mean reserves and somewhat stronger when evaluated at low debt level.

(ii) Reserves are not used. Our model predicts that once a country accumulates reserves and exits the crisis zone, it does not subsequently draw them down. The reason is that reserves serve primarily as a liquidity backstop that deters runs on government bonds—not as insurance to be deployed ex post. We assess this implication by asking whether countries draw down reserves during international crisis episodes, and whether those with larger reserve buffers at the onset of a crisis experience systematically larger subsequent drawdowns.

Figure 8 plots the change in reserves during crisis episodes against the level of reserves held at the onset of the 2008–09 Global Financial Crisis and the 2020–21 COVID pandemic. Each dot represents a country-crisis pair. The figure shows that the majority (57%) of the country-crisis pairs feature an increase in reserves by the end of the crisis.[40] In addition, the estimated relationship is effectively flat, and the associated p-value of 0.22 indicates no statistically significant correlation between initial reserve levels and reserve usage during these two crisis episodes. These features contrast with existing models of reserves, which emphasize an insurance role of reserves (see, e.g., Bianchi et al., 2018). In models of that class, higher initial reserve stocks should be drawn down in crises, and larger initial buffers would allow for a larger drawdown to smooth out the crisis. Instead, the data align with our model's prediction: once accumulated, reserves are not used.[41]

Takeaway. Taken together, the evidence suggests that joint increases in debt and reserves are associated with reductions in spreads and that countries do not systematically draw down reserves in times of fiscal stress. This empirical pattern is consistent with our model and contrasts with the insurance role of reserves in buffer-stock models, which predict reserve drawdowns during crisis episodes and an increase in spreads from higher gross positions.

[40] Among the country-crisis pairs with negative changes in reserves, one-third show decreases in reserves smaller than 1% of GDP in absolute value.

[41] Formally, we estimate

$$\Delta \text{Reserves}_i = \text{intercept} + \beta_{\text{Res}} \, \text{Reserves}_{i,0} + \epsilon_i,$$

where $\Delta\text{Reserves}_i$ is the change in reserves over the crisis window for the country-crisis pair i, $\text{Reserves}_{i,0}$ is the reserve ratio for that country at the onset of the crisis, and ϵ_i is a random error. Buffer-stock models predict $\beta_{\text{Res}} < 0$ and $\mathbb{E}[\Delta\text{Reserves}_i] < 0$; we find $\hat{\beta}_{\text{Res}} \approx 0$ (insignificant) and a positive mass of $\Delta\text{Reserves}_i$, consistent with no on-path use of reserves. A residualized version of this relationship—conditioning on contemporaneous crisis controls and excluding extreme observations—is presented in Appendix I for robustness.

Table 2: Panel regressions

Dep. Variable:	Spread (in bps)			
	Full Sample		Low Debt	High Debt
	(1)	(2)	(3)	(4)
Reserves	-13.51***	-10.72***	-10.28***	-18.99***
	(1.29)	(1.83)	(2.97)	(4.14)
Sov.Debt	6.85***	8.73***	10.29***	10.71***
	(0.91)	(1.27)	(2.84)	(2.68)
Reserves × Sov.Debt		-0.11**	-0.38***	0.09
		(0.05)	(0.11)	(0.10)
Marg. Effect (in bps)	-6.65**	-5.01	-10.19*	-5.20
[p-value]	[0.043]	[0.204]	[0.084]	[0.451]
Num. Obs.	1,767	1,767	949	818
R2 Adj.	0.298	0.299	0.376	0.286

Note: All specifications include year dummies, country FEs, and additional macroeconomic controls. Reserves and Sov.Debt are measured as percentages of GDP. Robust standard errors are in parentheses. *p<0.1; **p<0.05; ***p<0.01. The "Marg. Effect" row reports the discrete marginal effect (on spreads) of increasing reserves and debt by 1% of GDP (see footnote 39).

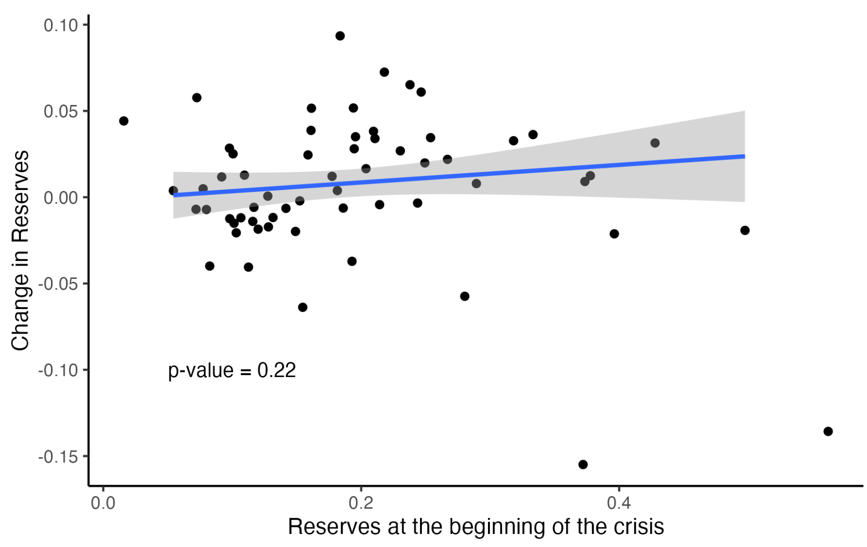

Figure 8: Reserves not used during crises. Each dot represents a country-crisis observation. The figure also shows the line of best fit, together with the 95% confidence interval.

4 Conclusions

This paper studies a government subject to rollover-crisis risk and asks whether it should accumulate reserves or reduce debt to become less vulnerable. The theory implies a non-monotonic adjustment: the government first deleverages—reducing debt and running down any initial reserves—then increases borrowing to accumulate reserves as it approaches the safe zone. Importantly, this debt-financed reserve accumulation lowers sovereign spreads by reducing vulnerability to rollover crises.

Our findings speak to central bank policy discussions on the appropriate level of international reserves (e.g., IMF, 2016, 2023). Following a debt crisis, a standard prescription is that countries accumulate reserves to improve their liquidity position. Our results show, however, that holding reserves is not optimal at the beginning of a deleveraging process. Rather, a highly indebted government should first reduce its debt and postpone the accumulation of reserves until the increase in reserves makes the government safe from a rollover crisis.

Our analysis suggests several directions for future research. For example, introducing a time-varying probability of rollover crises and fundamental shocks would be valuable to quantitatively examine the dynamics of international reserves, government debt, and spreads. Another promising avenue is to study the joint management of international reserves and the maturity structure of public debt.

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A Proofs

In this appendix, we prove the propositions laid out in the main text. In Appendix B we collect and prove several auxiliary results used in the proofs below.

A.1 Proof of Lemma 1

Proof. Consider first the case where the government cannot roll over the debt. Fix $a \geq 0$. Given that $\phi > 0$, we have that $V_R^-(a, 0) > V_D(a)$.

In addition, we have that consumption satisfies

$$\begin{aligned} c &= y + a - \left(\frac{\delta + r}{1 + r}\right) b - \frac{a'}{1 + r} + q(a', b')(b' - (1 - \delta)b) \\ &\leq y + a - \left(\frac{\delta + r}{1 + r}\right) b, \end{aligned}$$

where the inequality follows from $a' \geq 0$ and $b' \leq (1 - \delta)b$, and $q(a', b') \geq 0$.

For $b \geq (y + a)\left(\frac{1+r}{\delta+r}\right)$, it follows that $c \leq 0$. Therefore,

$$V_R^-(a, b) \leq \underline{u} + \beta \frac{\bar{u}}{1 - \beta} < \frac{u(y) - \phi}{1 - \beta} \leq V_D(a),$$

where the first inequality follows by the fact that the continuation value cannot be larger than $\bar{u}/(1 - \beta)$ and second inequality follows from Assumption 1.

From Lemmas B.1 and B.2, $V_R^-(b, a)$ is continuous and decreasing in b. It follows by the intermediate value theorem, that we can find a value $b^-(a) \in \left(0, (y + a)\left(\frac{1+r}{\delta+r}\right)\right)$, such that $V_R^-(a, b^-(a)) = V_D(a)$.

The proof for $b^+(a)$ is analogous. Finally, to see that $b^+(a) \geq b^-(a)$, note

$$V_R^+(a, b^-(a)) \geq V_R^-(a, b^-(a)) = V_D(a) = V_R^+(a, b^+(a)).$$

where the first inequality follows from the fact that $V_R^+(a, b) \geq V_R^-(a, b)$, the second follows by the definition of $b^-(a)$, while the third follows from the definition of $b^+(a)$. The result that $b^+(a) \geq b^-(a)$ follows from the fact that V_R^+ is decreasing in debt as shown in Lemma B.1.

□

A.2 Proof of Proposition 1

Proof. Assume toward a contradiction that the optimal sequence is such that $c_t > c_{t+1}$ for some t. We will show that there is an alternative policy that delivers a strict welfare improvement. Let $c(\cdot), a'(\cdot), b'(\cdot)$ correspond to the Markov equilibrium policies and consider the following alternative policy:

(i) For $\tau = 0, 1, \ldots, t-1$, $a_{\tau+1} = a'(a_\tau, b_\tau)$, $b_{\tau+1} = b'(a_\tau, b_\tau)$, and $c_\tau = c(a_\tau, b_\tau)$;

(ii) For $\tau = t$, $\tilde{a}_{\tau+1} = a'(a_t, b_t)$, $\tilde{b}_{\tau+1} = b'(a_t, b_t) - \frac{\varepsilon}{q(a_{t+1}, b_{t+1})}$, and

$$\tilde{c}_\tau(a_\tau, b_\tau) = y + a_t - \left(\frac{\delta + r}{1+r}\right) b_t - \frac{1}{1+r} a_{\tau+1} + q(\tilde{a}_{\tau+1}, \tilde{b}_{\tau+1})(\tilde{b}_{\tau+1} - (1-\delta) b_t)$$

with $\varepsilon > 0$;

(iii) For $\tau > t$, $\tilde{b}_{\tau+1} = b'(\tilde{a}_\tau, \tilde{b}_\tau)$, $\tilde{a}_{\tau+1} = a'(\tilde{a}_\tau, \tilde{b}_\tau)$, and $\tilde{c}_\tau = c(\tilde{a}_\tau, \tilde{b}_\tau)$.

Notice that the alternative policy satisfies the budget constraint in period t and thus is feasible. This alternative policy yields expected lifetime utility

$$\begin{aligned}\tilde{U} &= u(\tilde{c}_t) + \beta(1-\lambda) V_R^+(\tilde{a}_{t+1}, \tilde{b}_{t+1}) + \beta \lambda V_D(\tilde{a}_{t+1}) \\ &\geq u(c_t - \varepsilon) + \beta(1-\lambda) V_R^+(a_{t+1}, \tilde{b}_{t+1}) + \beta \lambda V_D(a_{t+1}),\end{aligned}$$

where we used that $\tilde{a}_{t+1} = a_{t+1}$. Consumption in t under this alternative policy is

$$\begin{aligned}\tilde{c}_t(a_t, b_t) &= c(a_t, b_t) - \varepsilon + [q(a_{t+1}, \tilde{b}_{t+1}) - q(a_{t+1}, b_{t+1})](\tilde{b}_{t+1} - (1-\delta) b_t) \\ &= c(a_t, b_t) - \varepsilon,\end{aligned}$$

where the second equality follows from Lemma B.4. The original policy yields

$$U = u(c_t) + \beta(1-\lambda) V_R^+(a_{t+1}, b_{t+1}) + \beta \lambda V_D(a_{t+1}).$$

Let $\Delta U \equiv \tilde{U} - U$ denote the increase in lifetime utility from following the alternative policy. Using the expressions above, we have that

$$\frac{\Delta U}{\varepsilon} = \frac{[u(c_t - \varepsilon) - u(c_t)]}{\varepsilon} + \beta(1-\lambda) \frac{\left[V_R^+\left(a_{t+1}, b_{t+1} - \frac{\varepsilon}{q(a_{t+1}, b_{t+1})}\right) - V_R^+(a_{t+1}, b_{t+1})\right]}{\varepsilon}.$$

We have

$$\lim_{\varepsilon \downarrow 0} \frac{u(c_t - \varepsilon) - u(c_t)}{\varepsilon} = -u'(c_t),$$

and with the envelope condition,

$$\lim_{\varepsilon\downarrow 0}\frac{V_R^+\left(a_{t+1}, b_{t+1}-\frac{\varepsilon}{q(a_{t+1},b_{t+1})}\right)-V_R^+(a_{t+1},b_{t+1})}{\varepsilon}=$$

$$\frac{u'(c_{t+1})}{q(a_{t+1},b_{t+1})}\left[\left(\frac{\delta+r}{1+r}\right)+(1-\delta)q(a_{t+2},b_{t+2})\right],$$

where we use that, by Lemma B.4, starting from an optimal policy, a marginal reduction in debt does not change the bond price.

In addition, the bond price equation (4) implies

$$q(a_{t+1},b_{t+1})=\frac{(1-\lambda)}{1+r}\left[\left(\frac{\delta+r}{1+r}\right)+(1-\delta)q(a_{t+2},b_{t+2})\right].$$

Replacing this expression above and using $\beta(1+r)=1$, we obtain

$$\lim_{\varepsilon\downarrow 0}\frac{\Delta U}{\varepsilon}=-u'(c_t)+u'(c_{t+1})>0$$

where the strict inequality follows from the assumption that $c_t > c_{t+1}$ and the strict concavity of the utility function. Hence, the deviation with small $\varepsilon > 0$ strictly improves utility. That is, an allocation where consumption is strictly decreasing for some t cannot be optimal.

□

A.3 Proof of Lemma 2

Proof. **Part (i).** Note that if $(0,(1-\delta)b)\in\mathbf{S}$, then any portfolio (a',b') respecting $a'\geq 0$ and $b'\leq(1-\delta)b$ will be in the safe zone, which follows from the monotonicity of $b^-(a)$ shown in Proposition 2. The government's problem is therefore

$$V_R^-(a,b)=\max_{a'\geq 0,b'\leq(1-\delta)b}\left\{u\left(y+a-b-\frac{a'-b'}{1+r}\right)+\beta\,V_S\left(a'-b'\right)\right\}.$$

where V_S is given by (9). Given that the problem is strictly concave, optimality requires

$$u'\left(y+a-b-\frac{a'-b'}{1+r}\right)\geq\beta(1+r)\,u'\left(y+(1-\beta)\big(a'-b'\big)\right),$$

with equality if $a'>0$ and $b'<(1-\delta)b$. Using $\beta(1+r)=1$, we obtain that an interior solution requires $a'-b'=a-b$. This implies a contradiction since $a'=b'+a-b\leq(1-\delta)b+a-b<0$ since $a<\delta b$. Therefore, the solution must be at the corner with $a'=0$

and $b' = (1-\delta)b$.

Part (ii). We know that $V_R^-(a,b) \leq V_R^+(a,b) \leq V_S(a-b)$, from Lemma B.5. In addition, note that it is feasible to set $c = y + (1-\beta)(a-b) = c_S(a-b)$, $a' - b' = a - b$, with $(a', b') \in \mathbf{S}$, $a' \geq 0$, and $b' \leq (1-\delta)b$. Because this yields $V_S(a-b)$, this must be optimal.

The final part of the proof refers to the comparison of $b^-(a)$ and $b^+(a)$, as defined in (6) and (7). To prove the stated relationships, it suffices to look at initial portfolios at the b^- boundary, because of the strict monotonicity of the value functions. If $a \geq \delta b^-(a)$ and $(a - \delta b^-(a), (1-\delta)b^-(a)) \in S$, then the policy $(a', b') = (a - \delta b^-(a), (1-\delta)b^-(a))$ is feasible with and without a run and yields a safe continuation value with constant consumption $c_S(a - b^-(a))$. Hence $V_R^-(a, b^-(a)) = V_R^+(a, b^-(a))$, and therefore $b^-(a) = b^+(a)$.

For the other cases, notice first that $V_R^+(a, b^-(a)) = V_S(a - b^-(a))$, by Lemma B.5, because $(a, b^-(a)) \in \mathbf{S}$. If either (i) $a < \delta b^-(a)$, or (ii) $a \geq \delta b^-(a)$ but $(a - \delta b^-(a), (1-\delta)b^-(a)) \in \mathbf{C}$, the government cannot achieve the safe value when facing a run, and so we have $V_R^-(a, b^-(a)) < V_S(a - b^-(a))$. Therefore, for these other cases, we have $V_R^-(a, b^-(a)) < V_R^+(a, b^-(a))$. Since $V_R^+(a,b)$ is decreasing in b, it follows that $b^-(a) < b^+(a)$.

□

A.4 Proof of Proposition 2

Proof. Let $c_R^+(a,b)$ be the solution for consumption from (3) and let $c_D(a) \equiv y + (1-\beta)a$ denote the constant default consumption associated with (1). Since $(a, b^+(a)) \in \mathbf{C}$, Lemma B.7 implies

$$c_R^+(a, b^+(a)) < c_D(a)\,.$$

Pick any $\varepsilon \in \Big(0,\, c_D(a) - c_R^+(a, b^+(a))\Big)$. Consider the state $(a + \varepsilon, b^+(a))$ in the repayment problem (3). We have that

$$V_R^+(a+\varepsilon, b^+(a)) \geq u\Big(c_R^+(a, b^+(a)) + \varepsilon\Big) + \beta \mathbb{E} V\left(a'\left(a, b^+(a)\right), b'\left(a, b^+(a)\right)\right)$$

and

$$V_R^+(a, b^+(a)) = u\Big(c_R^+(a, b^+(a))\Big) + \beta \mathbb{E} V(a'(a, b^+(a), b'(a, b^+(a)))\,.$$

Combining these two, we obtain

$$V_R^+(a+\varepsilon, b^+(a)) \;\geq\; V_R^+(a, b^+(a)) + \Big[u\Big(c_R^+(a, b^+(a)) + \varepsilon\Big) - u\Big(c_R^+(a, b^+(a))\Big)\Big]\,. \tag{A.1}$$

Using (10), we also have

$$V_D(a+\varepsilon) = V_D(a) + \frac{u\Big(c_D(a) + (1-\beta)\varepsilon\Big) - u\Big(c_D(a)\Big)}{1-\beta}\,. \tag{A.2}$$

Combining (A.1)–(A.2), we obtain

$$V_R^+(a+\varepsilon, b^+(a)) - V_D(a+\varepsilon) \geq \tag{A.3}$$
$$\Big(u(c_R^+ + \varepsilon) - u(c_R^+)\Big) - \frac{u(c_D + (1-\beta)\varepsilon) - u(c_D)}{1-\beta},$$

where we write $c_R^+ = c_R^+(a, b^+(a))$ and $c_D = c_D(a)$ for brevity.

By the mean value theorem, there exist $\xi_R \in (c_R^+, c_R^+ + \varepsilon)$ and $\xi_D \in (c_D, c_D + (1-\beta)\varepsilon)$ such that

$$u(c_R^+ + \varepsilon) - u(c_R^+) = u'(\xi_R)\varepsilon \quad \text{and} \quad \frac{u(c_D + (1-\beta)\varepsilon) - u(c_D)}{1-\beta} = u'(\xi_D)\varepsilon\,.$$

Since $\varepsilon < c_D - c_R^+$, we have $\xi_R < c_D < \xi_D$; hence, $\xi_R < \xi_D$. Strict concavity of $u(\cdot)$ implies that $u'(\xi_R) > u'(\xi_D)$. Therefore, the right-hand side of (A.3) is strictly positive, implying

$$V_R^+(a+\varepsilon, b^+(a)) > V_D(a+\varepsilon)\,.$$

Using the definition of $b^+(a+\varepsilon)$, we have

$$V_R^+(a+\varepsilon, b^+(a)) > V_D(a+\varepsilon) = V_R^+(a+\varepsilon, b^+(a+\varepsilon))\,.$$

Since $V_R^+(a,b)$ is strictly decreasing in b, by Lemma B.1, we obtain

$$b^+(a+\varepsilon) \;>\; b^+(a)\,.$$

Since this holds for any a, this implies that $b^+(a)$ is strictly increasing in a.

The proof for b^- is analogous where $c_R^-(a,b) < c_D(a)$ follows from Lemma B.8. □

A.5 Proof of Proposition 4

Proof. With $\delta = 1$, the value of repayment in a run becomes

$$V_R^-(a,b) = \max_{a' \geq 0} u\left(y + a - b - \frac{a'}{1+r}\right) + \beta V_S(a'),$$

where we have used that the government is safe tomorrow after repaying all the debt. The solution for a' is given by $a' = \max\{0, a - b\}$. If $a \geq b$, then we have $V_R^-(a, b) = V_S(a - b)$ and hence the government is not in the crisis zone. Considering therefore $a < b$, we obtain that the government is in the safe zone if and only if

$$u(y + a - b) + \frac{\beta}{1 - \beta} u(y) \geq \frac{u(y + a(1 - \beta)) - \phi}{1 - \beta}. \tag{A.4}$$

The lowest-NFA safe portfolio solves

$$\min_{a \geq 0, b} \; a - b,$$
$$\text{subject to (A.4)}$$

Since the objective is decreasing in b, the constraint must bind. Solving out b from the constraint and replacing it in the objective, the problem can then be rewritten as

$$\min_{a \geq 0} u^{-1} \left(\frac{u\left(y + (1 - \beta)a\right) - \phi - \beta u(y)}{1 - \beta} \right) - y.$$

Since $u(\cdot)^{-1}$ is strictly increasing and continuous, we have that the objective is strictly increasing in a. Therefore, the minimum is attained at zero, namely $a^\star = 0$.

With $\delta = 0$, the value of repaying in a run becomes $V_R^-(a, b) = \frac{u(y + (a - b)(1 - \beta))}{1 - \beta}$ and the lowest-NFA safe portfolio solves

$$\min_{a \geq 0, b} \; a - b,$$
$$\text{subject to}$$
$$u(y + (a - b)(1 - \beta)) = \; u(y + a(1 - \beta)) - \phi \,.$$

Solving out b from the constraint and replacing in the objective, the problem can be rewritten as

$$\min_{a \geq 0} \frac{u^{-1}\left(u(y + (1 - \beta)a) - \phi\right) - y}{1 - \beta}.$$

Since $u(\cdot)^{-1}$ is strictly increasing and continuous, we have that the objective is strictly increasing in a. Therefore, the minimum is attained at zero, namely $a^\star = 0$. □

A.6 Proof of Proposition 5

Proof. The proof is by construction. Fix $y > 0$ and $\beta \in (0,1)$ and set $r = \beta^{-1} - 1$ so that $\beta(1+r) = 1$. Pick x such that

$$y < x < \frac{y(1+r)}{r},$$

and define

$$\Delta \equiv \frac{y(1+r)}{x} - r.$$

The bounds on x imply $\Delta \in (0,1)$.

In the first step, for any $\delta \in (0,\Delta)$, we define $\phi = \phi(x,\delta)$ by the indifference condition $V_R^-(0,x) = V_D(0)$:

$$\phi(x,\delta) = u(y) - \left[(1-\beta)u\left(y - \left(\frac{\delta+r}{1+r}\right)x\right) + \beta u\left(y - (1-\beta)(1-\delta)x\right)\right].$$

For $x \in \left(0, \frac{y(1+r)}{\delta+r}\right)$ this function is well-defined and satisfies $\phi(x,\delta) > 0$.

In the second step, we define

$$F(\delta,x) \equiv \beta(1-\delta)\left[u'\left(y - \left(\frac{\delta+r}{1+r}\right)x\right) - u'\left(y - (1-\beta)(1-\delta)x\right)\right] - u'(y).$$

Condition (15) is equivalent to having $F(\delta,x) > 0$. We have

$$F(0,x) = -u'(y) < 0.$$

In addition, we have

$$\lim_{\delta\uparrow\Delta} F(\delta,x) = +\infty.$$

The second result follows because

$$\lim_{\delta\uparrow\Delta} y - \left(\frac{\delta+r}{1+r}\right)x = 0,$$

and the Inada condition.

By continuity, there exists $\varepsilon > 0$ such that $F(\delta,x) > 0$ for all $\delta \in (\Delta-\varepsilon,\Delta) \subset (0,1)$. Choose any such δ and set $\phi = \phi(x,\delta)$. Then condition (15) holds and Proposition 3 implies $a^\star > 0$. □

A.7 Proof of Proposition 6

The proof has two parts.

Proof. **Part (i).** Suppose the government chooses $(\tilde{a}, \tilde{b}) \in \mathbf{S}$ such that $(\tilde{a}, \tilde{b}) \neq (a^\star, b^\star)$. We can show that the utility is then lower. To see this, we need to show that

$$u\left(y + a - b + \left(\frac{\tilde{b} - \tilde{a}}{1+r}\right)\right) + \frac{\beta}{1-\beta} u\left(y + (1-\beta)(\tilde{a} - \tilde{b})\right) < u\left(y + a - b + \left(\frac{b^\star - a^\star}{1+r}\right)\right) + \frac{\beta}{1-\beta} u\left(y + (1-\beta)(a^\star - b^\star)\right). \quad \text{(A.5)}$$

Define the following objects:

$$\Delta \equiv \frac{b^\star - a^\star}{1+r} - \frac{\tilde{b} - \tilde{a}}{1+r} > 0,$$
$$\hat{c} \equiv y + a - b + \left(\frac{b^\star - a^\star}{1+r}\right) < y + (1-\beta)(a^\star - b^\star) \equiv c^\star.$$

where the first and second inequality follow respectively from the definition of $(a^\star, b^\star)$ and $a - b < a^\star - b^\star$. Using these expressions, we can rewrite (A.5) as

$$u(\hat{c} - \Delta) + \frac{\beta}{1-\beta} u\left(c^\star + \left(\frac{1-\beta}{\beta}\right)\Delta\right) < u(\hat{c}) + \frac{\beta}{1-\beta} u\left(c^\star\right).$$

Rearranging, we need to show that the following holds:

$$u(\hat{c}) - u(\hat{c} - \Delta) > \frac{\beta}{1-\beta}\left[u\left(c^\star + \left(\frac{1-\beta}{\beta}\right)\Delta\right) - u(c^\star)\right]. \quad \text{(A.6)}$$

The result follows from an application of the mean-value theorem, the strict concavity of $u(\cdot)$, and the fact that $\hat{c} < c^\star$. Namely, there exists $x \in (\hat{c} - \Delta, \hat{c})$, such that

$$u'(x)\Delta = u(\hat{c}) - u(\hat{c} - \Delta). \quad \text{(A.7)}$$

Similarly, there exists $z \in \left(c^\star, c^\star + \left(\frac{1-\beta}{\beta}\right)\Delta\right)$, such that

$$u'(z)\Delta = \frac{u\left(c^\star + \left(\frac{1-\beta}{\beta}\right)\Delta\right) - u(c^\star)}{(1-\beta)/\beta}. \quad \text{(A.8)}$$

Since $z > c^\star > \hat{c} > x, \Delta > 0$ and $u(\cdot)$ is strictly concave, we have that $u'(x) > u'(z)$. Using this strict inequality and rearranging (A.7) and (A.8), we obtain (A.6), as we wanted to show.

□

Proof. **Part (ii).** Suppose, toward a contradiction, that $a' > 0$. Consider an alternative portfolio given by

$$(\tilde{a}, \tilde{b}) = \left(a' - \varepsilon, b' - \frac{\varepsilon(1-\lambda)}{q(a',b')(1+r)}\right),$$

where $\varepsilon > 0$.

We first show that for small ε, the bond price remains the same. There are two cases to consider. If the government's original portfolio (a', b') is in the interior of an iso-T region, then by continuity, it is straightforward to show that it can reduce reserves and debt and remain in the same iso-T region. If the government is at the boundary between two iso-T regions, Auxiliary Lemma B.9 shows that this alternative portfolio with lower reserves and lower debt moves the government into the interior of the same iso-T region, keeping the bond price unchanged.

With this alternative policy $(\tilde{a}, \tilde{b})$, consumption in period t is therefore given by

$$\begin{aligned}\tilde{c}_R^+(a,b) &= c_R^+(a,b) + \frac{\varepsilon}{1+r} - q(a',b')\frac{\varepsilon(1-\lambda)}{q(a',b')(1+r)} \\ &= c_R^+(a,b) + \frac{\lambda\varepsilon}{1+r}.\end{aligned}$$

Notice that there is a drop in the continuation value under default in $t+1$, since the alternative portfolio has lower reserves $(a' > a' - \varepsilon)$. The change in lifetime welfare induced by this alternative portfolio, which we denote by ΔU, is given by

$$\begin{aligned}\Delta U = &\left[u(\tilde{c}_R^+(a,b)) - u(c_R^+(a,b))\right] + \beta\lambda\left[V_D(a'-\varepsilon) - V_D(a')\right] \\ &+ \beta(1-\lambda)\left[V_R^+(\tilde{a},\tilde{b}) - V_R^+(a',b')\right].\end{aligned} \tag{A.9}$$

Equivalently, with $\varepsilon > 0$,

$$\begin{aligned}\frac{\Delta U}{\varepsilon} = &\frac{\left[u(\tilde{c}_R^+(a,b)) - u(c_R^+(a,b))\right]}{\varepsilon} + \beta\lambda\frac{\left[V_D(a'-\varepsilon) - V_D(a')\right]}{\varepsilon} \\ &+ \beta(1-\lambda)\frac{\left[V_R^+(\tilde{a},\tilde{b}) - V_R^+(a',b')\right]}{\varepsilon}.\end{aligned}$$

We want to establish that $\lim_{\varepsilon\downarrow 0}\frac{\Delta U}{\varepsilon} > 0$. We have that

$$\lim_{\varepsilon\downarrow 0}\frac{\left[u(c_R^+(a,b)+\frac{\lambda\varepsilon}{1+r})-u(c_R^+(a,b))\right]}{\varepsilon} = u'(c_R^+(a,b))\frac{\lambda}{1+r}\,, \tag{A.10}$$

and, from the envelope condition,

$$\lim_{\varepsilon\downarrow 0}\frac{V_D(a'-\varepsilon)-V_D(a')}{\varepsilon} = -u'(c_D(a')). \tag{A.11}$$

As established in Lemma B.9, we have

$$\lim_{\varepsilon\downarrow 0}\frac{\left[V_R^+(\tilde{a},\tilde{b})-V_R^+(a',b')\right]}{\varepsilon} = 0\,. \tag{A.12}$$

Plugging (A.10), (A.11) and (A.12) into (A.9), and using $\beta(1+r)=1$, we arrive at:

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{\Delta U}{\varepsilon} &= \frac{\lambda}{1+r}\left[u'(c_R^+(a,b))-u'(c_D(a'))\right]\\ &\geq \frac{\lambda}{1+r}\left[u'(c_R^+(a',b'))-u'(c_D(a'))\right]\\ &> 0\,.\end{aligned}$$

The first inequality follows from the strict concavity of the $u(\cdot)$ and from Proposition 1, which establishes $c_R^+(a,b)\leq c_R^+(a',b')$. The last inequality uses again the strict concavity of $u(\cdot)$ and the fact that $c_R^+(a',b') < c_D(a')$, a result established in Auxiliary Lemma B.7.

This deviation is profitable, contradicting the optimality of (a',b'). Therefore, we conclude that $a'=0$. □

A.8 Proof of Corollary 1

Proof. Lemma B.10 establishes that if the initial portfolio is $(a,b)\in\mathbf{C}$ with $a-b<a^\star-b^\star$, then along the deleveraging process, the government never chooses a portfolio with NFA higher than $a^\star-b^\star$. Moreover, part (ii) of Proposition 6 indicates that the government chooses $a_{t+1}=0$ for all $t<T$ (i.e., as long as it remains in the crisis zone). Finally, part (i) of Proposition 6 indicates that the government exits the crisis zone by choosing $(a^\star,b^\star)$. Once the government reaches the safe zone, it stays in the safe zone. □

A.9 Proof of Proposition 7

Proof. If the government picks $(a^\star, b + a^\star - a)$, then the government is safe next period. This follows from the fact that $b + a^\star - a \leq b^\star$ and that $b^-(a)$ is increasing in a, as shown in Proposition 2. In addition, notice that with such a policy, the stationary consumption is attained, that is $c = y + (1 - \beta)(a - b)$, and the value of choosing this portfolio is given by $V_S(a - b)$. Given that $V_R^+(a, b) \leq V_S(a - b)$ (Lemma B.5, part i), it thus follows that the portfolio $(a^\star, b + a^\star - a)$ achieves the optimal solution. □

Online Appendix to "International Reserve Management under Rollover Crises"

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B Auxiliary Lemmas

Here we state and prove a number of auxiliary lemmas that we use in Appendix A.

Lemma B.1 (Monotonicity of value functions). *The value functions $V_R^+(a,b)$ and $V_R^-(a,b)$ are strictly decreasing in debt. Moreover, $V_R^+(a,b)$, $V_R^-(a,b)$ and $V_D(a)$ are strictly increasing in reserves.*

Proof. Fix a and let $b_1 < b_2$. Let (a_2', b_2') be optimal at (a, b_2), and let c_2 be implied by the budget constraint. Consider state (a, b_1) and choose

$$(a_1', b_1') = (a_2', b_2')\,.$$

The budget constraint implies

$$c_1 = c_2 + \left(\frac{\delta + r}{1+r} + q(a_2', b_2')(1-\delta)\right)(b_2 - b_1) > c_2\,,$$

since $b_2 - b_1 > 0$, $q(a_2', b_2') \geq 0$. Because $u(\cdot)$ is strictly increasing and the continuation value is the same, we have

$$V_R^+(a, b_1) > V_R^+(a, b_2)\,.$$

The other monotonicity results follow analogously. □

Lemma B.2 (Continuity). *The value functions V_R^+, V_R^- are continuous in (a,b) in the feasible domain set.*

Proof. Consider the continuity of V_R^+. Take two states (a^A, b^A) and (a^B, b^B) where the constraint set is non-empty. For any candidate continuation portfolio (a', b') traded at price $q = q(a', b')$, define the current-consumption implied by the budget constraint as

$$\tilde{C}(a, b; a', b', q) \equiv y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + q\Big(b' - (1-\delta)b\Big).$$

For any fixed (a', b', q) we have

$$\tilde{C}(a^A, b^A; a', b', q) - \tilde{C}(a^B, b^B; a', b', q) = (a^A - a^B) - \left(\left(\frac{\delta + r}{1 + r}\right) + (1 - \delta)q\right)(b^A - b^B).$$

Hence,

$$\left|\tilde{C}(a^A, b^A; a', b', q) - \tilde{C}(a^B, b^B; a', b', q)\right| \le |a^A - a^B| + \left(\left(\frac{\delta + r}{1 + r}\right) + (1 - \delta)q\right)|b^A - b^B|.$$

Using $q \le (1 + r)^{-1}$,

$$\left|\tilde{C}(a^A, b^A; a', b', q) - \tilde{C}(a^B, b^B; a', b', q)\right| \le |a^A - a^B| + |b^A - b^B|. \tag{B.1}$$

Let $(a^{A'}, b^{A'})$ and $(a^{B'}, b^{B'})$ be optimal continuation choices at (a^A, b^A) and (a^B, b^B), respectively. Let $q^{A'}$ and $q^{B'}$ denote the equilibrium prices at which these optimal policies trade. To save notation, $\tilde{C}^{i,j}$ be the consumption under state (a^i, b^i) and choice-price pair $(a^{j'}, b^{j'}, q^{j'})$, that is, $\tilde{C}^{A,B'} \equiv \tilde{C}(a^A, b^A; a^{B'}, b^{B'}, q^{B'})$. Then optimality implies

$$\begin{aligned} V_R^+(a^A, b^A) &= u\left(\tilde{C}^{A,A'}\right) + \beta \mathbb{E} V(a^{A'}, b^{A'}) \\ &\ge u\left(\tilde{C}^{A,B'}\right) + \beta \mathbb{E} V(a^{B'}, b^{B'}), \end{aligned} \tag{B.2}$$

and similarly

$$\begin{aligned} V_R^+(a^B, b^B) &= u\left(\tilde{C}^{B,B'}\right) + \beta \mathbb{E} V(a^{B'}, b^{B'}) \\ &\ge u\left(\tilde{C}^{B,A'}\right) + \beta \mathbb{E} V(a^{A'}, b^{A'}). \end{aligned} \tag{B.3}$$

Subtract (B.3) from $V_R^+(a^A, b^A) = u(\tilde{C}^{A,A'}) + \beta \mathbb{E} V(a^{A'}, b^{A'})$ to get the upper bound

$$V_R^+(a^A, b^A) - V_R^+(a^B, b^B) \le u\left(\tilde{C}^{A,A'}\right) - u\left(\tilde{C}^{B,A'}\right). \tag{B.4}$$

Likewise, subtract the identity for $V_R^+(a^B, b^B)$ from (B.2) to get the lower bound

$$V_R^+(a^A, b^A) - V_R^+(a^B, b^B) \ge u\left(\tilde{C}^{A,B'}\right) - u\left(\tilde{C}^{B,B'}\right). \tag{B.5}$$

Combining (B.4)–(B.5),

$$u\left(\tilde{C}^{A,A'}\right) - u\left(\tilde{C}^{B,A'}\right) \ge V_R^+(a^A, b^A) - V_R^+(a^B, b^B) \ge u\left(\tilde{C}^{A,B'}\right) - u\left(\tilde{C}^{B,B'}\right). \tag{B.6}$$

Concavity of $u(\cdot)$ implies that for all $c, c' \geq 0$,

$$|u(c') - u(c)| \leq u(|c' - c|) - \underline{u}. \tag{B.7}$$

Define $\omega(\Delta) \equiv u(\Delta) - u(0)$. Since u is continuous at 0 and $u(0)$ is finite, we have $\omega(\Delta) \to 0$ as $\Delta \downarrow 0$.

By (B.1), we have $|\widetilde{C}_{A,A'} - \widetilde{C}_{B,A'}| \leq \Delta$ and $|\widetilde{C}_{A,B'} - \widetilde{C}_{B,B'}| \leq \Delta$. Applying (B.7) to both utility differences in (B.6) gives

$$\left|V_R^+(a^A, b^A) - V_R^+(a^B, b^B)\right| \leq \omega(\Delta) = u(\Delta) - u(0).$$

Letting $(a^B, b^B) \to (a^A, b^A)$ implies $\Delta \to 0$ and thus the right-hand side converges to 0. Hence V_R^+ is continuous in (a, b).

The proof of continuity for V_R^- can be established analogously, after defining

$$\tilde{C}^-(a, b; a', q) \equiv y + a - \left(\frac{\delta + r}{1 + r}\right) b - \frac{a'}{1 + r}.$$

which yields $|\widetilde{C}^-(a^A, b^A; a') - \widetilde{C}^-(a^B, b^B; a')| \leq |a^A - a^B| + |b^A - b^B|$, and repeating steps above. □

Lemma B.3 (Positive issuances in the crisis zone)**.** *Consider a portfolio* $(a, b) \in \mathbf{C}$. *Then, a government that can roll over the debt chooses* $b' > (1 - \delta)b$.

Proof. By construction, when $(a, b) \in \mathbf{C}$, we have $V_R^+(a, b) > V_R^-(a, b)$. Suppose the optimal choice in (3) is (a', b') with $b' \leq (1 - \delta)b$. Then, (a', b') also solve (5) this would imply that a government that cannot rollover the debt would be able to replicate the portfolio of the government that can rollover, which means that $V_R^-(a, b) \geq V_R^+(a, b)$. Since $V_R^+(a, b) \geq V_R^-(a, b)$, this would imply $V_R^+(a, b) = V_R^-(a, b)$, contradicting that the government is in the crisis zone. □

Lemma B.4. *Let* $(a, b) \in \mathbf{C}$ *and let* (a', b') *denote the optimal policies, such that* $(a', b') \in \mathbf{C}$. *Then, for sufficiently small* $\varepsilon > 0$*, we have* $q(a', b' - \varepsilon) = q(a', b')$.

Proof. Assume, by way of contradiction, that for any $\varepsilon > 0$, $\Delta_q(\varepsilon) \equiv q(a', b' - \varepsilon) - q(a', b') >$

0. Consumption under the original policies (a', b', c) is:

$$c = y + a - \left(\frac{\delta + r}{1 + r}\right) b - \frac{a'}{1 + r} + q(a', b')(b' - (1 - \delta)b).$$

Consider an alternative policy to be such that the chosen portfolio is $(a', b' - \varepsilon)$. Under this alternative policy, consumption is:

$$\begin{aligned} c_\varepsilon &= y + a - \left(\frac{\delta + r}{1 + r}\right) b - \frac{a'}{1 + r} + q(a', b' - \varepsilon)(b' - \varepsilon - (1 - \delta)b) \\ &= c - \varepsilon q(a', b' - \varepsilon) + \Delta_q(\varepsilon)(b' - (1 - \delta)b). \end{aligned}$$

Observe that $q(\cdot)$ is a step function because, for fixed a', the equilibrium price depends only on which iso-T region the continuation portfolio (a', b') belongs to, and these regions are separated by threshold values of b'. Hence $q(a', b')$ is constant on intervals of b' and can change only when b' crosses a boundary. As a result, there exists $\varepsilon_m > 0$ such that for all $\varepsilon \leq \varepsilon_m$, $\Delta_q(\varepsilon)$ is invariant in ε, that is $\Delta_q(\varepsilon) = \Delta_q > 0$.

Since $(a, b) \in \mathbf{C}$ implies $b' - (1 - \delta)b > 0$, (by Lemma B.3), we can define the threshold

$$\bar{\varepsilon} \equiv \frac{\Delta_q(b' - (1 - \delta)b)}{\Delta_q + q(a', b')},$$

such that any $\varepsilon \in (0, \min\{\bar{\varepsilon}, \varepsilon_m\})$ delivers a consumption level strictly higher than under the original plan, $c_\varepsilon > c$. Since $u(\cdot)$ is strictly increasing and $V_R^+(a', b')$ is strictly decreasing in b' (Lemma B.1), it follows that

$$u(c_\varepsilon) + \beta V_R^+(a', b' - \varepsilon) > u(c) + \beta V_R^+(a', b'),$$

contradicting the optimality of (a', b', c). □

Lemma B.5. *Let $(a, b) \in \mathbf{C}$. The following is true:*

(i) $V_R^+(a, b) \leq V_S(a - b)$, *and*

(ii) If there exists $(a', b') \in \mathbf{S}$, *such that* $a' - b' = a - b$, *then* $V_R^+(a, b) = V_S(a - b)$ *and* $c_R^+(a, b) = c_S(a - b)$.

The proof has two parts.

Proof. **Part (i).** Fix $(a, b) \in \mathbf{C}$, and let $\{c_t, a_{t+1}, b_{t+1}\}_{t\geq 0}$ denote the equilibrium allocations conditional on repayment and a good sunspot (rollover). For any given t, the optimal portfolio can take the government into either $\mathbf{C}$ or $\mathbf{S}$ in $t+1$. In the former case (i.e., $(a_{t+1}, b_{t+1}) \in \mathbf{C}$), the following holds:

$$V_R^+(a_{t+1}, b_{t+1}) \geq (1-\lambda)V_R^+(a_{t+1}, b_{t+1}) + \lambda V_D(a_{t+1}) = \mathbb{E}V(a_{t+1}, b_{t+1}, \zeta_{t+1}) .$$

In the latter case, when $(a_{t+1}, b_{t+1}) \in \mathbf{S}$, we have

$$\mathbb{E}V(a_{t+1}, b_{t+1}, \zeta_{t+1}) = V_S(a_{t+1} - b_{t+1}) = V_R^+(a_{t+1}, b_{t+1}) .$$

Thus, we can bound the period-t value under repayment as follows:

$$V_R^+(a_t, b_t) \leq u(c_t) + \beta V_R^+(a_{t+1}, b_{t+1}) .$$

Iterating forward, we get

$$V_R^+(a, b) \leq \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{B.8}$$

Next, note that when $(a_t, b_t) \in \mathbf{C}$, we have (by Lemma B.3) that the net issuance is positive, $b_{t+1} > (1-\delta)b_t$. Using that $q(a_{t+1}, b_{t+1}) \leq \frac{1}{1+r} = \beta$, the budget constraint at time t implies

$$c_t \leq y + (a_t - b_t) - \beta(a_{t+1} - b_{t+1}) .$$

Multiplying both sides by β^t and summing we get,

$$\sum_{t=0}^{\infty} \beta^t c_t \leq \frac{y}{1-\beta} + (a-b) ,$$

which is equivalent to

$$\sum_{t=0}^{\infty} (1-\beta)\beta^t c_t \leq y + (1-\beta)(a-b) . \tag{B.9}$$

Noticing that $\sum_{t=0}^{\infty}(1-\beta)\beta^t = 1$, we have

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \leq \frac{1}{1-\beta} u\left(\sum_{t=0}^{\infty}(1-\beta)\beta^t c_t\right) \leq \frac{u\ (y + (1-\beta)(a-b))}{1-\beta} . \tag{B.10}$$

The first inequality follows from the concavity of $u(\cdot)$, and the second inequality follows from equation (B.9) and the monotonicity of $u(\cdot)$. Combining (B.8) and (B.10), we arrive at $V_R^+(a, b) \leq V_S(a-b)$. $\square$

Proof. **Part (ii).** If there exists $(a', b') \in \mathbf{S}$ such that $a' - b' = a - b$, then picking such portfolio and $c_R^+(a, b) = c_S(a - b)$ yields $V_S(a - b)$, which implies $V_R^+(a, b) \geq V_S(a - b)$. By part (i) of the lemma, this must be optimal.

□

Lemma B.6. *The lowest-safe NFA portfolio* $(a^\star, b^\star) = \arg\min\{a - b : (a, b) \in \mathbf{S}\}$ *is such that* $a^\star < b^\star$*.*

Proof. By definition, $\forall (a, b) \in \mathbf{S}$, $a^\star - b^\star \leq a - b$. Since $(0, b^-(0)) \in \mathbf{S}$, we have

$$a^\star - b^\star \leq 0 - b^-(0) < 0\,,$$

and we conclude that $a^\star < b^\star$. □

Lemma B.7. *For any* $(a, b) \in \mathbf{C}$*, we must have that* $c_R^+(a, b) < c_D(a)$*.*

Proof. Fix $(a, b) \in \mathbf{C}$, and let $\{c_t, a_{t+1}, b_{t+1}\}_{t \geq 0}$ denote the equilibrium allocations conditional on repayment and a good sunspot (rollover). Let $T \geq 0$ be the first date such that the beginning-of-period portfolio is in the safe zone, $(a_T, b_T) \in \mathbf{S}$.

Along repayment, the budget constraint implies the following:

$$\begin{aligned} c_t &= y + a_t - \left(\frac{\delta + r}{1 + r}\right) b_t - \frac{a_{t+1}}{1 + r} + q_t(a_{t+1}, b_{t+1})(b_{t+1} - (1 - \delta) b_t) \\ &\leq y + a_t - \left(\frac{\delta + r}{1 + r}\right) b_t - \frac{a_{t+1}}{1 + r} + \frac{1}{1 + r}(b_{t+1} - (1 - \delta) b_t) \\ &\leq y + a_t - b_t - \frac{1}{1 + r}(a_{t+1} - b_{t+1})\,, \end{aligned}$$

where we used that $q_t(a_{t+1}, b_{t+1}) \leq \frac{1}{1+r}$ and, by Lemma B.3, $b_{t+1} > (1 - \delta) b_t$ given that $(a_t, b_t) \in \mathbf{C}$, and the government can rollover its debt.

Multiplying each time-t constraint above by β^t, summing from $t = 0$ to $t = T - 1$, using $\beta(1 + r) = 1$, we write

$$\sum_{t=0}^{T-1} \beta^t c_t \leq \sum_{t=0}^{T-1} \beta^t y + (a - b) - \beta^T (a_T - b_T)\,.$$

Since $(a_T, b_T) \in \mathbf{S}$, consumption from T onward is constant at $c_S = y + (1-\beta)(a_T - b_T)$, and so we can write

$$\sum_{t=T}^{\infty} \beta^t c_t = \beta^T \frac{c_S}{1-\beta} = \beta^T \left(\frac{y}{1-\beta} + a_T - b_T \right).$$

Adding the last two expressions gives

$$\sum_{t=0}^{\infty} \beta^t c_t \leq \frac{y}{1-\beta} + (a-b) = \frac{y + (1-\beta)(a-b)}{1-\beta}.$$

Using $c_D(a) = y + (1-\beta)a$ and recalling that Proposition 1 implies $c_R^+(a,b) \leq c_t$ for all t:

$$\frac{c_R^+(a,b)}{1-\beta} = \sum_{t=0}^{\infty} \beta^t c_R^+(a,b) \leq \sum_{t=0}^{\infty} \beta^t c_t \leq \frac{y + (1-\beta)(a-b)}{1-\beta} < \frac{y + (1-\beta)(a)}{1-\beta} = \frac{c_D(a)}{1-\beta}.$$

That is, we have $c_R^+(a,b) < c_D(a)$, as we wanted to show. □

Lemma B.8. *For any $(a,b) \in \mathbf{C}$, we must have $c_R^-(a,b) < c_D(a)$.*

Proof. It suffices to show that $c_R^-(a,b) \leq c_S(a-b)$ since $c_S(a-b) \equiv y + (1-\beta)(a-b) < y + (1-\beta)a \equiv c_D(a)$. Notice that, using $\beta(1+r) = 1$, we have:

$$\begin{aligned} c_R^-(a,b) &\equiv y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + q(a',b')\,(b' - (1-\delta)b) \\ &\leq c_S(a-b) + \beta(a - \delta b - a'), \end{aligned}$$

where the inequality follows from $q(a',b') \geq 0$ and $b' \leq (1-\delta)b$ under V_R^- and $(a,b) \in \mathbf{C}$.

First, consider the case $a \leq \delta b$. Since $a' \geq 0$, we have $a - \delta b - a' \leq 0$ and $c_R^-(a,b) \leq c_S(a-b)$, and the result is shown.

If $a > \delta b$ and if $(a - \delta b, (1-\delta)b) \in \mathbf{S}$, then from Lemma 2 we have that consumption is given by $c_S(a-b)$, and the result is shown.

Finally, if $a > \delta b$ and if $(a - \delta b, (1-\delta)b) \notin \mathbf{S}$, then the first-order condition for a', together with the envelope condition, when $(a', b') \notin \mathbf{S}$ (that is, $(a', b') \in \mathbf{S}$), implies

$$\begin{aligned} u'(c_R^-(a,b)) &\geq (1-\lambda)u'(c_R^+(a',b')) + \lambda u'(c_D(a')) \\ &> (1-\lambda)u'(c_D(a')) + \lambda u'(c_D(a')) \\ &= u'(c_D(a')), \end{aligned}$$

where the second inequality uses $c_R^+(a', b') < c_D(a')$ by Lemma B.7 and strict concavity of $u(\cdot)$. Next, assume, by way of contradiction, that $c_R^-(a, b) \geq c_D(a)$. By the strict concavity of $u(\cdot)$, we have

$$u'(c_D(a)) \geq u'(c_R^-(a, b)) > u'(c_D(a')) .$$

This requires $c_D(a) < c_D(a')$, and so $a' > a$. As a result, $0 > a - a' > a - \delta b - a'$. Therefore, $c_R^-(a, b) < c_S(a - b)$, which leads to $c_D(a) \leq c_R^-(a, b) < c_D(a)$, a contradiction. □

Lemma B.9. *Consider a government with initial portfolio $(a, b) \in \mathbf{C}$ that lies on the boundary of the iso-T region for $T = J$. That is, starting from (a, b), the sovereign is indifferent between exiting the crisis zone after J periods and after $J + 1$ periods. Then, there exists a small $\varepsilon > 0$ such that the alternative portfolio*

$$(\tilde{a}, \tilde{b}) \equiv \left(a - \varepsilon,\ b - \frac{\varepsilon(1 - \lambda)}{q(a, b)(1 + r)} \right)$$

lies in the interior of the iso-T region for $T = J$. Moreover,

$$\lim_{\varepsilon \downarrow 0} \frac{V_R^+(\tilde{a}, \tilde{b}) - V_R^+(a, b)}{\varepsilon} = 0.$$

Proof. Let $(a, b) \in \mathbf{C}$ be such that the government is indifferent between exiting the crisis zone in J or $J + 1$ periods: $V_R^+(a, b) = W^J(a, b) = W^{J+1}(a, b)$, where $W^T(a, b)$ denotes the sovereign's value under a plan that exit the crisis zone in T periods. Define

$$\Delta W = W^J \left(a - \varepsilon, b - \frac{\varepsilon(1 - \lambda)}{q(a, b)(1 + r)} \right) - W^{J+1} \left(a - \varepsilon, b - \frac{\varepsilon(1 - \lambda)}{q(a, b)(1 + r)} \right).$$

We want to argue that $\frac{\Delta W}{\varepsilon} > 0$ when ε is sufficiently small. In that case, at $(\tilde{a}, \tilde{b})$ the government strictly prefers to exit in J periods rather than $J + 1$, so $(\tilde{a}, \tilde{b})$ is in the interior of iso-T J, and shares the same price as the portfolio (a, b).

Using that $W^J(a, b) = W^{J+1}(a, b)$, we can write the above as

$$\frac{\Delta W}{\varepsilon} = \frac{\left[W^J \left(a - \varepsilon, b - \frac{\varepsilon(1-\lambda)}{q(a,b)(1+r)} \right) - W^J (a, b) \right]}{\varepsilon} - \frac{\left[W^{J+1} \left(a - \varepsilon, b - \frac{\varepsilon(1-\lambda)}{q(a,b)(1+r)} \right) - W^{J+1} (a, b) \right]}{\varepsilon} \tag{B.11}$$

Now, we have that

$$\lim_{\varepsilon \downarrow 0} \frac{W^J\left(a-\varepsilon, b-\frac{\varepsilon(1-\lambda)}{q(a,b)(1+r)}\right) - W^J(a,b)}{\varepsilon} = -\frac{\partial W^J(a,b)}{\partial a} - \frac{1-\lambda}{q(a,b)(1+r)}\frac{\partial W^J(a,b)}{\partial b}.$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{W^{J+1}\left(a-\varepsilon, b-\frac{\varepsilon(1-\lambda)}{q(a,b)(1+r)}\right) - W^{J+1}(a,b)}{\varepsilon} = \\ -\frac{\partial W^{J+1}(a,b)}{\partial a} - \frac{1-\lambda}{q(a,b)(1+r)}\frac{\partial W^{J+1}(a,b)}{\partial b}.$$

Replacing these last two equations and taking the limit in (B.11), we get

$$\lim_{\varepsilon \downarrow 0} \frac{\Delta W}{\varepsilon} = \left[-\frac{\partial W^J(a,b)}{\partial a} - \frac{1-\lambda}{q(a,b)(1+r)}\frac{\partial W^J(a,b)}{\partial b}\right] \\ - \left[-\frac{\partial W^{J+1}(a,b)}{\partial a} - \frac{1-\lambda}{q(a,b)(1+r)}\frac{\partial W^{J+1}(a,b)}{\partial b}\right].$$

From the envelope conditions, we have

$$\frac{\partial W^J(a,b)}{\partial a} = u'(c^J),$$
$$\frac{\partial W^J(a,b)}{\partial b} = -u'(c^J)\left[\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^J\right],$$
$$\frac{\partial W^{J+1}(a,b)}{\partial a} = u'(c^{J+1}),$$
$$\frac{\partial W^{J+1}(a,b)}{\partial b} = -u'(c^{J+1})\left[\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^{J+1}\right].$$

where c^T is the current consumption under repayment when exiting in T periods, and q^T the bond price when exiting in T periods, for $T \in \{J, J+1\}$. Replacing these envelopes and rearranging

$$\lim_{\varepsilon \downarrow 0} \frac{\Delta W}{\varepsilon} = u'(c^J)\left[-1 + \frac{1-\lambda}{q(a,b)(1+r)}\left(\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^J\right)\right] \\ + u'(c^{J+1})\left[1 - \frac{1-\lambda}{q(a,b)(1+r)}\left(\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^{J+1}\right)\right]. \tag{B.12}$$

The term in the first bracket is zero. To see why, recall that the price recursion implies that

$$q^{J+1} = \frac{1-\lambda}{1+r}\left[\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^J\right].$$

As argued in footnote 21, if the government starts with $(a, b) \in \mathbf{C}$ and is indifferent between exiting in J or $J+1$ periods, then it chooses to exit in J periods, and the equilibrium prices reflect that. Therefore, when that portfolio (a, b) was chosen, in the previous period, the government was exiting the crisis zone in $J+1$ periods, which means $q(a, b) = q^{J+1}$. As a result,

$$\frac{1-\lambda}{q(a,b)(1+r)} \equiv \frac{1-\lambda}{q^{J+1}(1+r)} = \left[\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^J\right]^{-1}. \tag{B.13}$$

As a result, we can write (B.12) as

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \frac{\Delta W}{\varepsilon} &= u'(c^{J+1})\left[1 - \frac{\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^{J+1}}{\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^J}\right] \\
&= u'(c^{J+1})\left[\frac{(1-\delta)(q^J - q^{J+1})}{\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^J}\right] > 0.
\end{aligned}$$

where the inequality follows from $q^J > q^{J+1}$. Therefore, we established that at $(\tilde{a}, \tilde{b})$ the government strictly prefers to exit the crisis zone in J rather than $J+1$ periods. As a result, $(\tilde{a}, \tilde{b})$ lies, in the next period, on the interior of the iso-T with $T = J$. Since equilibrium bond prices are constant within each iso-T region, it follows that $q(\tilde{a}, \tilde{b}) = q(a, b)$ for all $\varepsilon > 0$ sufficiently small.

Finally, we argue that $\lim_{\varepsilon\downarrow 0} \dfrac{V_R^+(\tilde{a}, \tilde{b}) - V_R^+(a, b)}{\varepsilon} = 0$. Since for a small ε the perturbation $(\tilde{a}, \tilde{b})$ remains in the interior of the same iso-T region, the equilibrium price and the optimal continuation policy are locally unchanged. Therefore, the envelope conditions apply for the one-sided directional derivative of V_R^+ along this perturbation (see Milgrom and Segal, 2002 for envelope theorems with directional derivatives). Using these envelope conditions, we arrive at

$$\begin{aligned}
\frac{\partial V_R^+(a,b)}{\partial a} &= u'(c_R^+(a,b)), \\
\frac{\partial V_R^+(a,b)}{\partial b} &= -u'(c_R^+(a,b))\left[\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^J\right].
\end{aligned}$$

Therefore, we have

$$\lim_{\varepsilon \downarrow 0} \frac{\left[V_R^+(\tilde{a}, \tilde{b}) - V_R^+(a,b)\right]}{\varepsilon} = u'(c_R^+(a,b)) \left[-1 + \frac{1-\lambda}{q(a,b)(1+r)} \left(\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^J\right)\right].$$

Using (B.13), we get that the term in brackets is zero. Therefore, we conclude that

$$\lim_{\varepsilon \downarrow 0} \frac{\left[V_R^+(\tilde{a}, \tilde{b}) - V_R^+(a,b)\right]}{\varepsilon} = 0.$$

□

Lemma B.10 (No over-saving)**.** *Let* $(a, b) \in \mathbf{C}$ *be such that* $a - b < a^\star - b^\star$*. Suppose that* $(a'(a, b), b'(a, b)) \in \mathbf{C}$*. Then,* $a'(a, b) - b'(a, b) < a^\star - b^\star$*.*

Proof. There are two cases to consider.

First, if $(a', b') \in \mathbf{C}$ and $a' - b' = a^\star - b^\star$, then choosing $a^\star - b^\star$ is strictly preferred. To see why, define the levels of current consumption under choices (a', b') and $(a^\star, b^\star)$ as:

$$c = y + a - \left(\frac{\delta+r}{1+r}\right) b - \frac{a'}{1+r} + \frac{1-\lambda}{1+r}(b' - (1-\delta)b),$$
$$c^\star = y + a - \left(\frac{\delta+r}{1+r}\right) b - \frac{a^\star}{1+r} + \frac{1}{1+r}(b^\star - (1-\delta)b).$$

Taking $c^\star - c$, we find

$$c^\star - c = \frac{a' - a^\star + b^\star - b'}{1+r} + \frac{\lambda}{1+r}(b' - (1-\delta)b) > 0.$$

The first term is zero by $a' - b' = a^\star - b^\star$. The second term is positive since for $(a, b) \in \mathbf{C}$, Lemma B.3 implies $b' > (1-\delta)b$. Notice that for the (a', b') we used the bond price $q(a', b') = \frac{1-\lambda}{1+r}$: this follows from Proposition 7, which established the government exits in one period if it starts with an NFA equal to $a^\star - b^\star$. Next, we consider the continuation values. Observe that $(a', b') \in \mathbf{C}$ implies

$$V_R^+(a', b') \geq V_D(a').$$

Next, by Lemma B.5 (part *ii*) we have that

$$V_R^+(a', b') = V_S(a^\star - b^\star).$$

Thus, using $V_S(a^\star - b^\star) \geq V_R^+(a', b')$ (by Lemma B.5) and $a^\star - b^\star = a' - b'$ we arrive at

$$V_S(a^\star - b^\star) = \lambda V_S(a^\star - b^\star) + (1-\lambda)V_S(a^\star - b^\star) \geq \lambda V_D(a') + (1-\lambda)V_R^+(a', b').$$

Therefore, the continuation under $(a^\star, b^\star)$ is weakly higher than under (a', b'). Because $c^\star > c$, the lifetime utility of under $(a^\star, b^\star)$ is strictly higher than under (a', b').

We next prove that $a' - b' > a^\star - b^\star$ cannot be an equilibrium choice. Toward a contradiction, suppose that $a' - b' > a^\star - b^\star$. First, we prove that, in this case, consumption is strictly increasing over time. To see why, define

$$\begin{aligned} c_0 &= y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + \frac{1-\lambda}{1+r}(b' - (1-\delta)b), \\ c_1 &= y + (1-\beta)(a' - b'). \end{aligned}$$

Next, observe that, using $\beta(1+r) = 1$,

$$\begin{aligned} c_0 &= y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + \frac{1-\lambda}{1+r}(b' - (1-\delta)b) \\ &< y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + \frac{1}{1+r}(b' - (1-\delta)b) \\ &= y + a - b + \frac{b' - a'}{1+r} \\ &< y + a' - b' + \frac{b' - a'}{1+r} \\ &= c_1. \end{aligned}$$

The first inequality uses the fact that in the crisis zone, it must be that $b' > (1-\delta)b$, following Lemma B.3. The second strict inequality uses $a' - b' > a - b$.

We know that under the policy (a', b'), Proposition 7 guarantees that the government exits the crisis zone in the next period. Thus, the bond price is $q(a', b') = \frac{1-\lambda}{1+r}$ and the lifetime utility of following this policy is

$$\begin{aligned} W =& u\left(y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + \frac{1-\lambda}{1+r}(b' - (1-\delta)b)\right) \\ &\qquad + \beta\lambda \frac{(u(y + (1-\beta)a') - \phi)}{1-\beta} + \beta(1-\lambda)\frac{u(y + (1-\beta)(a' - b'))}{1-\beta}. \end{aligned}$$

Consider an alternative portfolio in which $(a', b' + \varepsilon)$ for small $\varepsilon > 0$. Note that for sufficiently small $\varepsilon > 0$, we still have that $a' - b' - \varepsilon > a^\star - b^\star$. Thus, $q(a', b' + \varepsilon) = \frac{1-\lambda}{1+r}$ and lifetime utility is

$$\begin{aligned}\widetilde{W} =& u\left(y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + \frac{1-\lambda}{1+r}(b' + \varepsilon - (1-\delta)b)\right) \\ &\qquad + \beta\lambda\frac{(u(y + (1-\beta)a') - \phi)}{1-\beta} + \beta(1-\lambda)\frac{u(y + (1-\beta)(a' - b' - \varepsilon))}{1-\beta}.\end{aligned}$$

Therefore, we can write the lifetime utility gain as

$$\frac{\widetilde{W} - W}{\varepsilon} = \frac{u\left(c_0 + \frac{(1-\lambda)}{1+r}\varepsilon\right) - u(c_0)}{\varepsilon} + \beta(1-\lambda)\frac{[u(c_1 - (1-\beta)\varepsilon) - u(c_1)]}{(1-\beta)\varepsilon}.$$

Now, using $\beta(1+r) = 1$ and taking the limit, we find:

$$\lim_{\varepsilon \to 0} \frac{\widetilde{W} - W}{\varepsilon} = \left(\frac{1-\lambda}{1+r}\right)[u'(c_0) - u'(c_1)]$$

Since $c_1 > c_0$, the strict concavity of $u(\cdot)$ gives us that this limit is positive. Therefore, the feasible alternative is strictly better, contradicting the optimality of (a', b').

Putting these two pieces together, we conclude that if $(a'(a,b), b'(a,b)) \in \mathbf{C}$, we have $a'(a,b) - b'(a,b) < a^\star - b^\star$. □

C A Special Case for Proposition 3

In the main text, we specified sufficient conditions for $a^\star > 0$. In this appendix, we show a parametric condition under quadratic utility. There are two parts. First, we derive the threshold $b^-(0)$ as a function of the parameters $(y, r, \delta, \phi, \bar{c})$. Then, we substitute $b^-(0)$ into inequality (15). In what follows, we assume $\delta \in (0, 1)$, $r > 0$, and $\phi > 0$.
Let us assume that the utility function is quadratic, defined for $c \leq \bar{c}$:

$$u(c) = -\frac{1}{2}(c - \bar{c})^2,$$

with $y < \bar{c}$. Throughout, we assume that the satiation point $\bar{c}$ is sufficiently large so that all consumption levels generated by the allocations we consider satisfy $c < \bar{c}$, for the empirically relevant range of (a, b). We present the results first, and leave the detailed derivations for the following subsections.

The threshold is

$$b^-(0) = \frac{(1+r)\left[(y-\bar{c})r + \sqrt{(y-\bar{c})^2 r^2 + 2r(r+\delta^2)\phi}\right]}{r(r+\delta^2)} > 0.$$

The necessary and sufficient condition for $a^\star > 0$ is

$$\phi \; > \; \frac{(y-\bar{c})^2 r\left[\, r + 2\delta - \delta^2 \,\right]}{2\,(1-\delta)^2\,\delta^2} \equiv \underline{\phi} > 0\,.$$

C.1 Deriving $b^-(0)$

First, observe that the $b^-(0)$ equation is

$$u\left(y - \left(\frac{\delta + r}{1+r}\right) b^-(0)\right) + \beta \frac{u\left(y - (1-\beta)(1-\delta)b^-(0)\right)}{1-\beta} = \frac{u(y) - \phi}{1-\beta}.$$

Let $x \equiv b^-(0)$ to simplify on notation and substitute the utility function

$$-\frac{1}{2}\left(y - \frac{\delta+r}{1+r}x - \bar{c}\right)^2 - \frac{1}{2}\frac{\beta}{(1-\beta)}\left(y - (1-\beta)(1-\delta)x - \bar{c}\right)^2 = \frac{-\frac{1}{2}(y-\bar{c})^2 - \phi}{1-\beta}.$$

Multiplying by $-2(1-\beta)$ and simplifying, we get

$$(1-\beta)\left(y - \bar{c} - \frac{\delta+r}{1+r}x\right)^2 + \beta\left(y - \bar{c} - (1-\beta)(1-\delta)x\right)^2 = (y-\bar{c})^2 + 2\phi.$$

Letting $\hat{y} \equiv y - \bar{c}$ and using $\beta(1+r) = 1$, we find

$$\frac{r}{1+r}\left(\hat{y} - \frac{\delta + r}{1+r}x\right)^2 + \frac{1}{1+r}\left(\hat{y} - \frac{r}{1+r}(1-\delta)x\right)^2 = \hat{y}^2 + 2\phi.$$

After some algebra, we find

$$r(r+\delta^2)x^2 - 2\hat{y}r(1+r)x - 2\phi(1+r)^2 = 0.$$

Hence, $b^-(0)$ solves the quadratic equation, with the unique positive solution, using $\hat{y} = y - c$:

$$b^-(0) = \frac{(1+r)\left[(y-\bar{c})r + \sqrt{(y-\bar{c})^2 r^2 + 2r(r+\delta^2)\phi}\right]}{r(r+\delta^2)}. \tag{C.1}$$

C.2 Parametric Condition for $a^\star > 0$

Next, take the $b^-(0)$ from (C.1) and plug into equation (15):

$$\beta(1-\delta)\left[u'\left(y - \left(\frac{\delta + r}{1+r}\right)b^-(0)\right) - u'\left(y - (1-\beta)(1-\delta)b^-(0)\right)\right] > u'(y). \tag{C.2}$$

Observe that $u'(c) = -c + \bar{c} > 0$. Using $(\delta + r)/(1+r) = \delta\beta + 1 - \beta$:

$$u'\left(y - \left(\frac{\delta + r}{1+r}\right)b^-(0)\right) = -y + (\delta\beta + (1-\beta))b^-(0) + \bar{c},$$

and

$$u'\left(y - (1-\beta)(1-\delta)b^-(0)\right) = -y + ((1-\beta) - \delta(1-\beta))b^-(0) + \bar{c}.$$

After some algebra, condition (C.2) requires

$$b^-(0) > \frac{-y + \bar{c}}{\beta(1-\delta)\delta} \equiv \frac{(-y+\bar{c})(1+r)}{(1-\delta)\delta}$$

Using the formula we found at (C.1), and after some algebra, we arrive at

$$\phi \ > \ \frac{(y-\bar{c})^2\, r\, [\, r + 2\delta - \delta^2\,]}{2\,(1-\delta)^2\,\delta^2} \equiv \underline{\phi}$$

Notice that when $r > 0$, $\delta \in (0,1)$, $y < \bar{c}$, we must have $\underline{\phi} > 0$.

D A Condition for Staying in the Safe Zone

In this appendix, we analyze the conditions under which policy strategies where the government transitions from the safe zone or the crisis zone to the default zone are not part of a Markov equilibrium. That is, we will rule out strategies where the government finds it optimal to default with probability one in the next period.

To obtain a sufficient condition, we can focus on the case where the government is indifferent between receiving $V_S(a-b)$ and defaulting today.[42] That is, (a,b) satisfies

$$u(y+(1-\beta)(a-b)) = u(y+(1-\beta)a) - \phi\,. \tag{D.1}$$

Let us denote by $\hat{b}(a)$ the unique value that satisfies (D.1). It is straightforward to see that $\hat{b}(a)$ is positive, increasing in a, and independent of δ.

Consider the payoff for the government of repaying in the current period and defaulting in the next period. We denote the value of this "deviation" as $\widetilde{V}(a,b;\delta)$—making explicit that the value depends on the maturity:

$$\widetilde{V}(a,b;\delta) = \max_{a'\geq 0} u(c) + \beta V_D(a'), \tag{D.2}$$

$$\text{subject to}$$

$$c = y + a - \left(\frac{\delta+r}{1+r}\right) b - \frac{a'}{1+r},$$

where the budget constraint uses that the bond price in the current period is zero given that the government will default tomorrow.

We have the following proposition.

Proposition D.1. *For any $a>0$, we have that*

$$\widetilde{V}\Big(a,\hat{b}(a);\delta\Big) \leq V_S\Big(a-\hat{b}(a)\Big),$$

[42] To see why it is a sufficient condition, consider a case where the government is at the $b^-(a)$ boundary. Because $V_S(a-b) \geq V_R^-(a,b) = V_D(a)$, a policy of defaulting tomorrow could be preferable to defaulting today, but still weakly lower than $V^S(a-b)$.

for any $\delta \geq \tilde{\delta}(a)$ *where*

$$\tilde{\delta}(a) \equiv \frac{y + (1-\beta)\Big(a - (1-\beta)\hat{b}(a)\Big) - u^{-1}(\Psi)}{(1-\beta)\beta\hat{b}(a)},$$

and

$$\Psi \equiv \beta u\left(y + (1-\beta)a\right) + (1-\beta)u\left(y + (1-\beta)\Big(a - \hat{b}(a)\Big)\right)$$

and where $\tilde{\delta}(a) \in (0,1)$. *That is, the government does not transition into the default zone for maturity values such that* $\delta \geq \tilde{\delta}(a)$.

Moreover, if $a = 0$, *there is no value for* δ *under which the government would leave the safe zone. That is,* $\widetilde{V}(0, \hat{b}(0); \delta) \leq V_S(0 - \hat{b}(0))$ *for any* $\delta \geq 0$.

Proof. Fix $a > 0$. Define the value from a deviation as $\tilde{F}(\delta; a, \hat{b}(a)) \equiv \widetilde{V}(a, \hat{b}(a); \delta) - V_S(a - b)$.

First, if $a' \geq 0$ does not bind in problem (D.2), we obtain

$$\widetilde{V}(a, \hat{b}(a); \delta) = \frac{u\left(y + (1-\beta)\left(a - \left(\frac{\delta+r}{1+r}\right)\hat{b}(a)\right)\right) - \beta\phi}{1-\beta}.$$

Notice that if instead $a' \geq 0$, binds, the deviation value is lower than the interior expression (and thus equation D provides an upper bound on the gain). Therefore, showing $F(\delta; a, b) \leq 0$ is sufficient to rule out the deviation in either case.

Using (D.1) to substitute out ϕ, we obtain, after direct algebra, the following:

$$\tilde{\delta}(a) \equiv \frac{y + (1-\beta)\Big(a - (1-\beta)\hat{b}(a)\Big) - u^{-1}(\Psi)}{(1-\beta)\beta\hat{b}(a)},$$

where

$$\Psi \equiv \beta u\left(y + (1-\beta)a\right) + (1-\beta)u\left(y + (1-\beta)\Big(a - \hat{b}(a)\Big)\right).$$

Notice that $F(\delta; a, \hat{b}(a))$ is decreasing in δ, since $\widetilde{V}(a, \hat{b}(a); \delta)$ is strictly decreasing in δ while V_S is independent of δ. Therefore, for $\delta \geq \tilde{\delta}(a)$, $F(\delta; a, \hat{b}(a)) \leq 0$; that is, $\widetilde{V}\Big(a, \hat{b}(a); \delta\Big) \leq V_S\Big(a - \hat{b}(a)\Big)$.

Next, we show that $\tilde{\delta}(a) > 0$. Observe that because $\hat{b}(a)$ is strictly positive and $u(\cdot)$ is strictly increasing, we have

$$u\left(y+(1-\beta)\left(a-\hat{b}(a)\right)\right) < u\left(y+(1-\beta)a\right).$$

Since Ψ is a convex combination of these two values and $\beta \in (0,1)$, we obtain

$$u\left(y+(1-\beta)\left(a-\hat{b}(a)\right)\right) < \Psi < u\left(y+(1-\beta)a\right),$$

and by strict monotonicity of u^{-1},

$$y+(1-\beta)\left(a-\hat{b}(a)\right) < u^{-1}(\Psi) < y+(1-\beta)a. \tag{D.3}$$

Next, we denote with Φ the sum of the first two terms in the numerator of $\tilde{\delta}(a)$,

$$\Phi = y+(1-\beta)\left(a-(1-\beta)\hat{b}(a)\right) = \beta\left(y+(1-\beta)a\right) + (1-\beta)\left(y+(1-\beta)\left(a-\hat{b}(a)\right)\right).$$

By strict concavity of u, Jensen's inequality implies that

$$u(\Phi) > \beta u\left(y+(1-\beta)a\right) + (1-\beta)u\left(y+(1-\beta)\left(a-\hat{b}(a)\right)\right) = \Psi.$$

Since u is increasing, this implies $\Phi > u^{-1}(\Psi)$, so $\tilde{\delta}(a) > 0$.

Towards showing that $\tilde{\delta}(a) < 1$, observe that from (D.3) we have $u^{-1}(\Psi) > y + (1 - \beta)(a - \hat{b}(a))$, hence

$$\Phi - u^{-1}(\Psi) < \Phi - \left(y+(1-\beta)\left(a-\hat{b}(a)\right)\right) = (1-\beta)\beta\hat{b}(a),$$

where the last equality follows by direct algebra. Dividing by $(1-\beta)\beta\hat{b}(a) > 0$ gives $\tilde{\delta}(a) < 1$. Therefore, $\tilde{\delta}(a) \in (0,1)$, when $a > 0$.

Finally, we consider the case with $a = 0$. We can write

$$\begin{aligned}
\widetilde{V}(0,\hat{b}(0);\tilde{\delta}(0)) &= u\left(y - \left(\frac{\tilde{\delta}(0)+r}{1+r}\right)\hat{b}(0)\right) + \beta V_D(0) \\
&= V_D(0) \\
&= u(c_D(0)) - \phi + \beta V_D(0) \\
&= u(c_S(0-\hat{b}(0))) + \beta V_D(0),
\end{aligned}$$

where the second line uses $\widetilde{V}(0,\hat{b}(0);\delta) = V_D(0)$ at the threshold $\delta = \tilde{\delta}(0)$, the third line follows from the definition of $V_D(0)$, and the fourth line uses equation (D.1) to substitute out $u(c_D(0)) - \phi$ with $u(c_S(0 - \hat{b}(0)))$. Therefore, using $\beta(1+r) = 1$ we get to

$$y - \left(\frac{\tilde{\delta}(0) + r}{1+r}\right)\hat{b}(0) = y - \frac{r}{1+r}\hat{b}(0)$$

which solves as $\tilde{\delta}(0) = 0$. Finally, since $\widetilde{V}(0, \hat{b}(0); \delta)$ is strictly decreasing in δ, we get that for any $\delta \geq 0$, $\widetilde{V}(0, \hat{b}(0); \delta) \leq V_S(0 - \hat{b}(0))$. □

Numerical results. To provide a sense of the parameter values that ensure that the government does not leave the safe zone, we numerically compute the threshold value for δ over which the government optimally stays in the safe zone. For $a \geq \left(\frac{\delta+r}{1+r}\right)\hat{b}(a)$, the expression is given by $\tilde{\delta}$ in Proposition D.1. Otherwise, when the constraint binds, there is no closed-form expression for $\tilde{\delta}$. We solve for $\tilde{\delta}$ numerically, taking into account the possibility that the constraint binds.

Figure D.1 plots the threshold value δ as a function of initial reserves a under the benchmark calibration. Panel (a) reports comparative statics with respect to the default cost parameter ϕ, while panel (b) reports the corresponding exercise for the coefficient of risk aversion σ. We can see in the figure that across all the parametrization, leaving the safe zone requires extremely long maturities (i.e., very low values for δ).

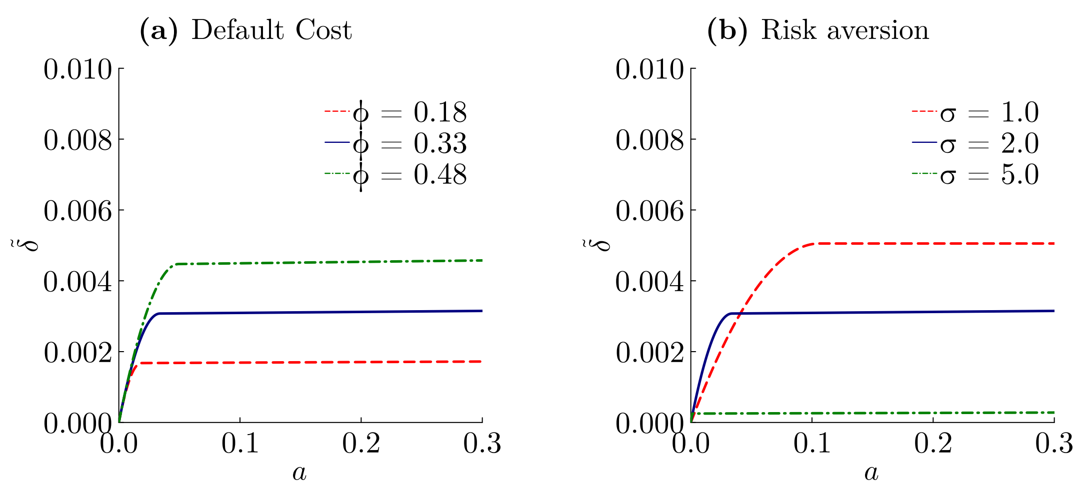

Figure D.1: Threshold $\tilde{\delta}$ as a function of initial reserves. The figure plots the threshold $\tilde{\delta}$ as a function of initial reserves a. Panel (a) varies the default cost parameter ϕ, and panel (b) varies the coefficient of risk aversion σ. In each panel, the blue solid line corresponds to the benchmark calibration. All remaining parameters are set as in Table 1.

E Additional Figures: Economy without Reserves

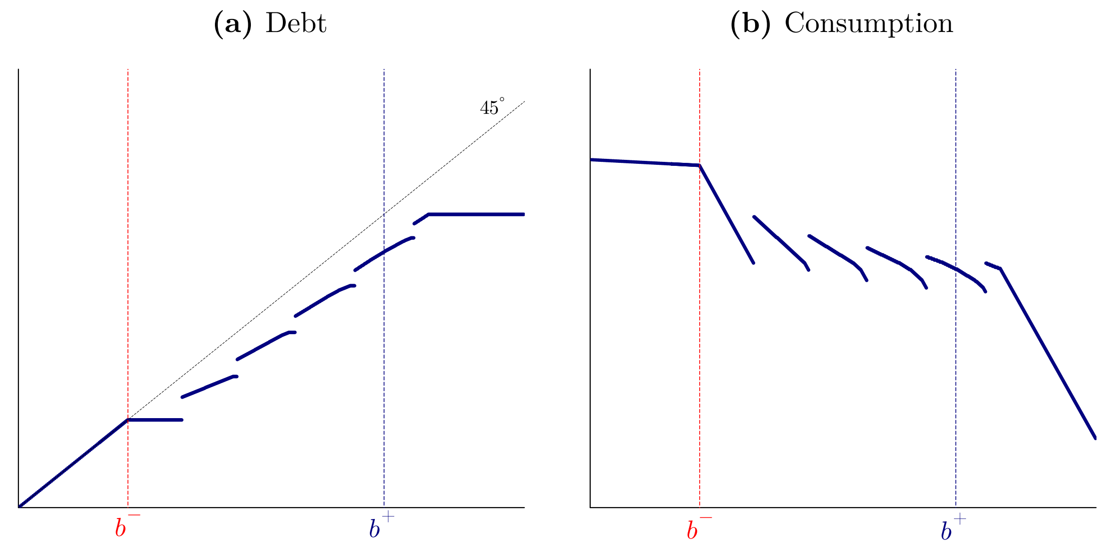

Figure E.1: Policy without reserves. The figure presents the policy functions for the economy when the government is restricted from accumulating reserves. Panel (a) shows the policy for debt, $b'(b)$. Panel (b) exhibits the policy for consumption, $c(b)$.

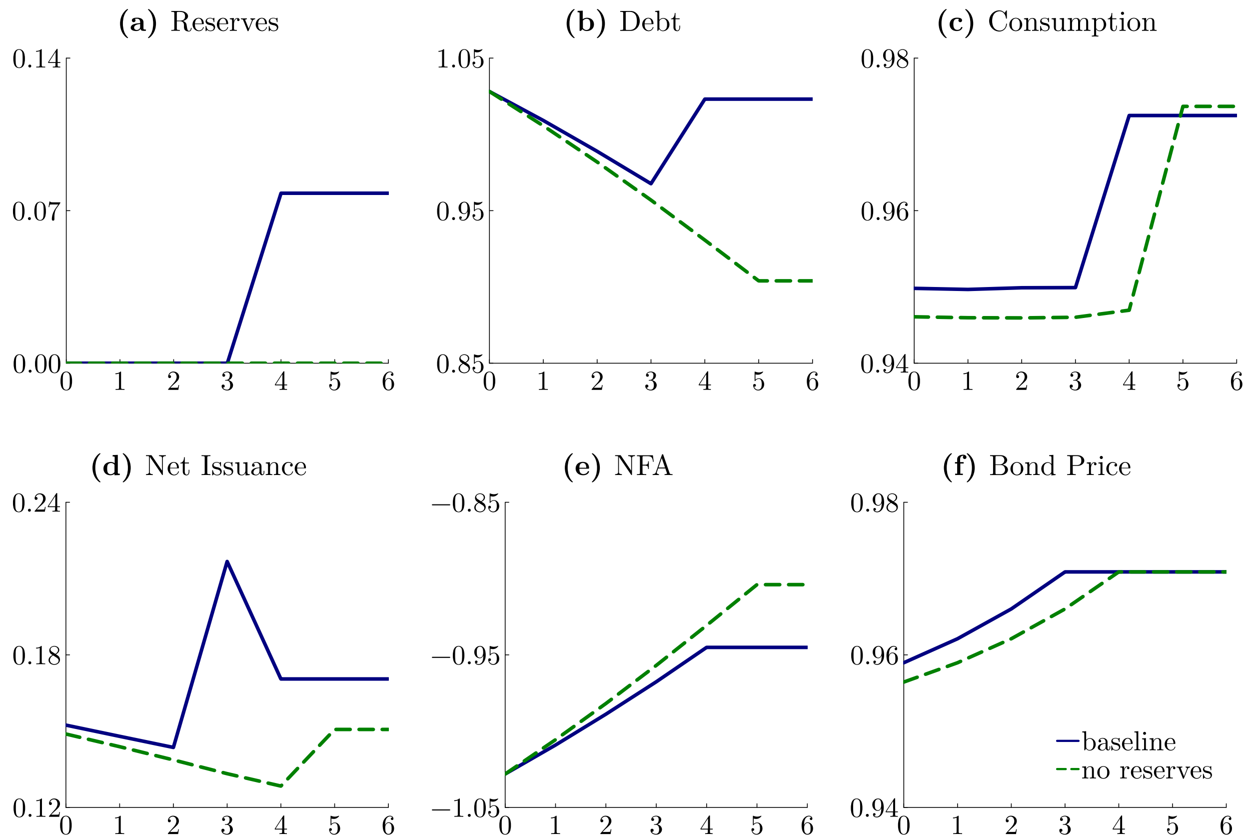

Figure E.2: Lower consumption and a longer time to exit without reserves. The government is assumed to start in the crisis zone with $b = 1.04$ and zero initial reserves.

F Varying the Probability of the Bad Sunspot, λ

In this appendix, we present a sensitivity analysis for λ, the probability that the economy is hit by the bad sunspot. Figure F.1 reports how four endogenous objects of interest vary with λ, keeping all other parameters fixed at their benchmark values (see Table 1).

Panel (a) of Figure F.1 shows that the government optimally takes longer to exit the crisis zone as λ decreases. If default is less likely, the government can sustain higher consumption by deleveraging more gradually. Panel (b) shows that a lower probability of the bad sunspot is associated with a higher $b^+(0)$. By definition, $b^+(0)$ solves the indifference condition $V_R^+(b,0) = V_D(0)$. A lower λ raises $V_R^+(b,0)$ at any given b by reducing the likelihood of a run, while $V_D(0)$ is unchanged; since $V_R^+(b,0)$ is decreasing in b (by Lemma B.1), the equality is restored only at a higher debt level, implying a higher $b^+(0)$.

Panels (c) and (d) of Figure F.1 show that $a^\star$ and $b^\star$ are invariant to λ under this parametrization. This arises because, with $(y, \sigma, \delta, r, c, \phi)$ fixed, we have both $(1-\delta)b^\star < b^-(0)$ and $a^\star - \delta b^\star < 0$, so a run at $(a^\star, b^\star)$ implies the chosen portfolio will be $(0, (1-\delta)b^\star)$, which is in the safe zone. Along the entire segment of the $b^-(a)$ boundary from $(0, b^-(0))$ up to $(a^\star, b^\star)$, facing a run will also lead to portfolios lying in the safe zone next period, so V_R^- depends only on V_S and is thereby independent of λ in this region.[43]

Figure F.2 reports the deleveraging dynamics for alternative values of λ. Consistent with the discussion above, a lower λ reduces the incentives to front-load the adjustment: the government exits the crisis zone more slowly, which implies more gradual deleveraging, higher consumption, and a more negative NFA position along the transition. Bond prices reflect two channels: first, conditional on being in the crisis zone in $t+1$, a higher λ lowers period-t bond prices by increasing the default probability; second, because a higher λ induces a faster exit, prices converge to the risk-free level in fewer periods.

[43]Appendix G presents an alternative parametrization under which the government remains vulnerable next period after repaying in a run, in which case $(a^\star, b^\star)$ vary with λ.

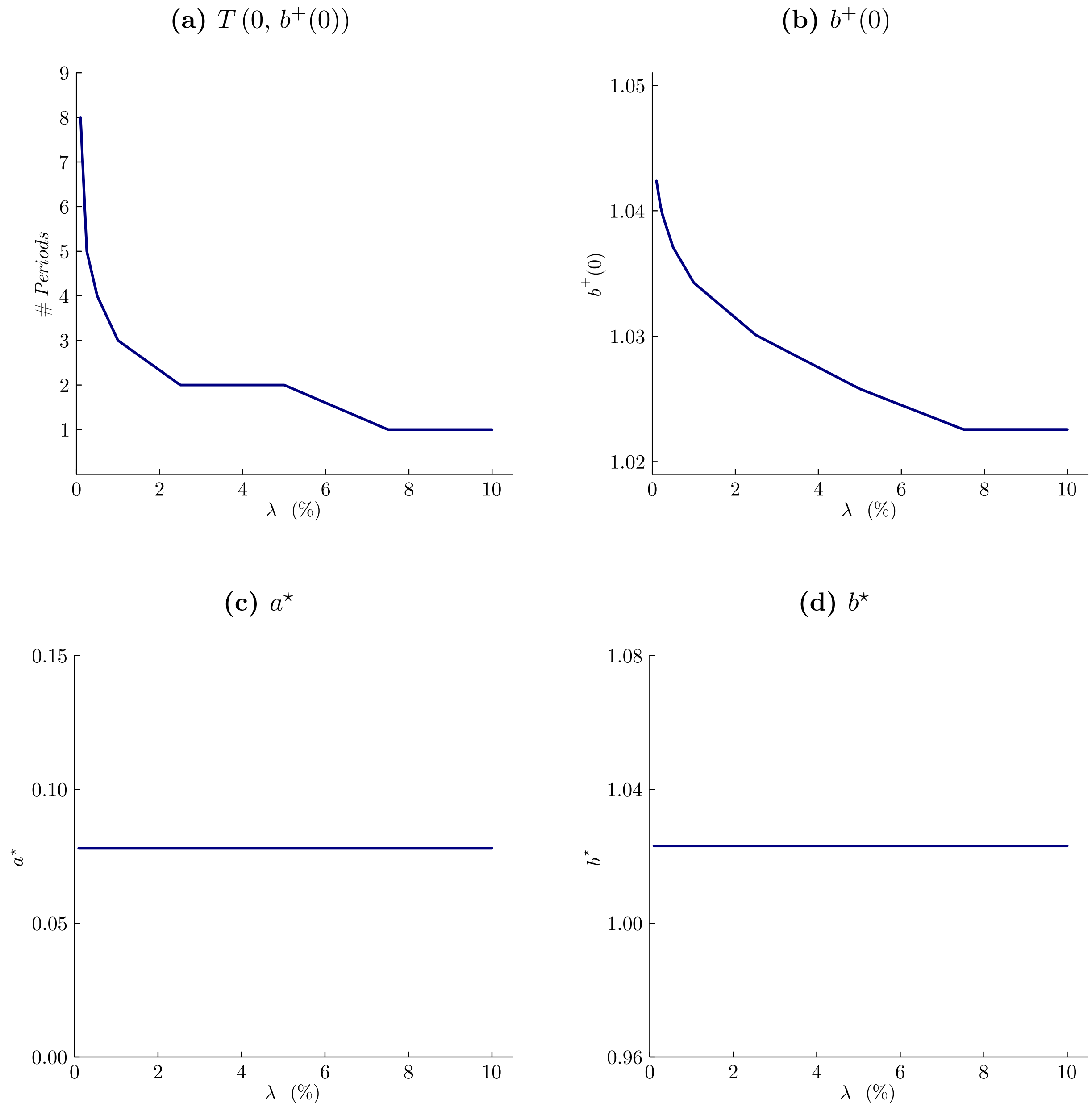

Figure F.1: Sensitivity to λ. Panel (a) shows the exit time $T(0, b^{+}(0))$, panel (b) $b^{+}(0)$, and panels (c) and (d) the lowest-NFA safe portfolio $a^{\star}$ and $b^{\star}$, for varying values of λ. All other parameters remain at their benchmark calibration. Note that the left-most point is $\lambda = 0.1\%$. The benchmark calibration has $\lambda = 0.5\%$.

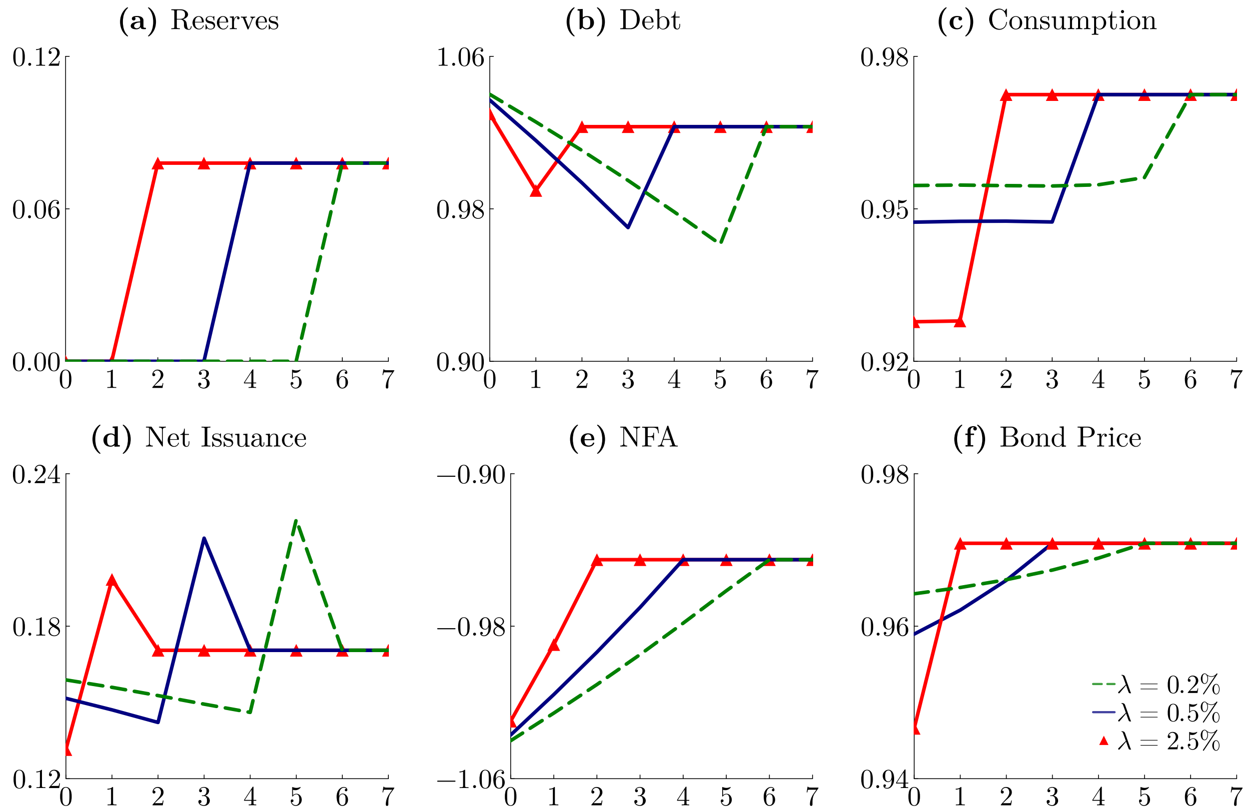

Figure F.2: Deleveraging dynamics for alternative λ. The government is assumed to start in the crisis zone with an initial portfolio $(0, b^+(0))$. Panels (a), (b), and (e) plot beginning-of-period levels of reserves, debt, and NFA, respectively. Notice that $b^+(0)$ is a function of λ. The blue solid line is for the baseline calibration, $\lambda = 0.5\%$. The green dashed line represents $\lambda = 0.2\%$, and the solid red line with triangles corresponds to $\lambda = 2.5\%$. All other parameters remain at their benchmark calibration in Table 1.

G Vulnerability In the Period After a Run

In this appendix, we present a configuration under which the government remains vulnerable in the next period after repaying the coupons due during a run. The logic is that even though the government starts from lower debt tomorrow, the necessary reduction in reserves to repay may end up leaving the government still vulnerable.

Figure G.1 illustrates this situation. Suppose the government starts at the initial portfolio $(a_0, b_0) \in \mathbf{C}$. Point A represents the portfolio tomorrow if the government pays the coupons, and chooses $b' = (1-\delta)b_0$, and $a' = 0$. At the portfolio $(0, (1-\delta)b_0)$, the government may be in the safe zone (as shown in panel [a]) or not (panels [b] and [c]). In the latter case, the government could choose $(0, (1-\delta)b_0)$ and remain exposed next period or change the portfolio (reducing debt or increasing reserves) to be safe next period. Under the baseline parametric conditions where $a^\star > 0$ and $b^\star > (1-\delta)b_0$, the change in the portfolio that requires the lower cut in consumption to reach the safe zone is increasing reserves to point $\mathrm{B} = (\bar{a}, (1-\delta)b_0)$, with $\bar{a}$ satisfying $b^-(\bar{a}) = (1-\delta)b_0$. Then, the government chooses point B (with $a' > 0$) if

$$u\left(y + a_0 - \left(\frac{\delta + r}{1+r}\right) b_0 - \frac{\bar{a}}{1+r}\right) + \beta V_S(\bar{a} - (1-\delta)b_0) \geq$$
$$u\left(y + a_0 - \left(\frac{\delta + r}{1+r}\right) b_0\right) + \beta \left[(1-\lambda) V_R^+(0, (1-\delta)b_0) + \lambda V_D(0)\right],$$

and point A (with $a' = 0$) otherwise.

Panel (b) in Figure G.1 presents a parametrization for which the government prefers to stay in point A, and risk being hit with the bad sunspot in the next period. Panel (c) shows a different parametrization, with a higher probability for the bad sunspot, in which the government prefers point B, reaching the safe zone in the current period.

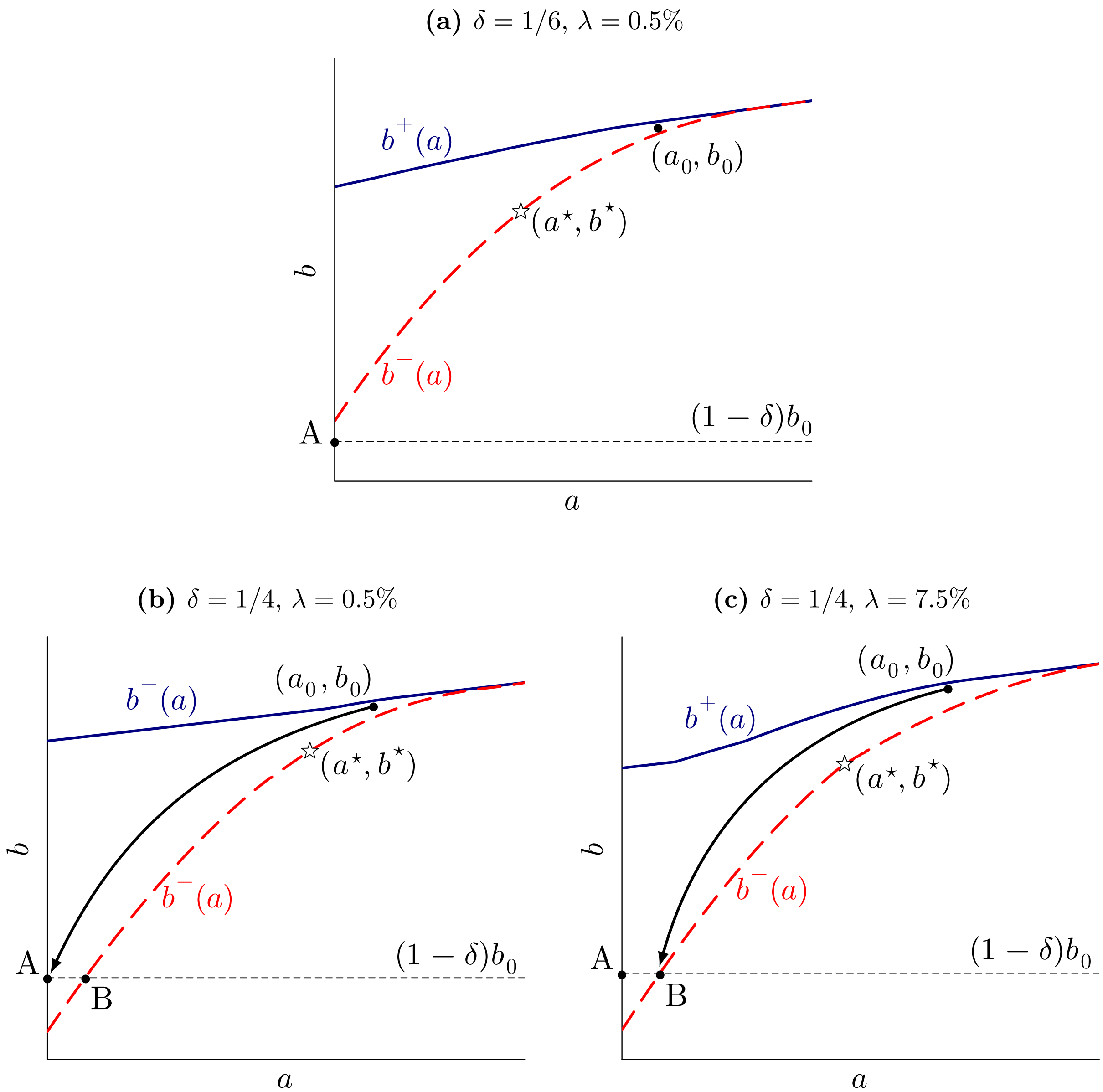

Figure G.1: Every panel shows the two boundaries, b^- and b^+, along with an initial portfolio $(a_0, b_0) \in \mathbf{C}$ and the lowest-NFA safe portfolio $(a^\star, b^\star)$. Point A is $(0, (1-\delta)b_0)$, which lies in the safe zone in panel (a) and in the crisis zone in panels (b) and (c). Point B, only in panels (b) and (c), denotes a candidate portfolio choice $(\bar{a}, b^-(\bar{a})) \in \mathbf{S}$. The panel titles indicate which parameter values (if any) are different from the benchmark calibration in Table 1.

H Numerical Algorithm

1. Set parameters $\{y, \sigma, \phi, \underline{c}, r, \delta, \lambda\}$. Set $\beta = (1+r)^{-1}$. Using bisection, compute $b^-(0)$ as the zero of the following function:

$$F(x) = u(y) - \phi - (1-\beta)u\left(y - \left(\frac{\delta + r}{1+r}\right)x\right) - \beta u\left(y - (1-\beta)(1-\delta)x\right)$$

   where recall that since $(0, b^-(0)) \in \mathbf{S}$, the continuation value after repaying coupons in a run starting from that portfolio is V_S.

2. Set a linearly spaced grid for reserves, $\mathcal{A}$, with $a_{\min} = 0.0$, $a_{\max} = 0.35$, and $N_a = 351$.

3. Set a linearly spaced grid for debt, $\mathcal{B}$.[44]

4. For each $a \in \mathcal{A}$, compute $V_D(a) = \frac{u(y+(1-\beta)a)-\phi}{1-\beta}$.

5. Set an iterator counter j to 1. For each value in the grid $(a, b) \in \mathcal{A} \times \mathcal{B}$, initialize the following guesses: $q_1(a, b) = \frac{1}{1+r}$, $V^+_{R,1}(a, b) = V^-_{R,1}(a, b) = \frac{u(y-(1-\beta)b_{\max})}{1-\beta}$, and $b^-_1(a) = b^+_1(a) = \infty$.

6. Given the price schedule q_j, solve for the value and policy functions under repayment. That is, for each $(a, b) \in \mathcal{A} \times \mathcal{B}$, go over all possible pairs $(\tilde{a}, \tilde{b}) \in \mathcal{A} \times \mathcal{B}$, searching for the optimal value as follows:

   (a) If the government can issue new debt, V^+_R:

      i. If (a, b) satisfies $b \leq b^-_j(a)$, then government is in the safe zone; i.e., $(a, b) \in \mathbf{S}$. Set $(a'_j, b'_j) = (a, b)$ and $V^+_{R,j+1}(a, b) = V_S(a - b)$.

      ii. If (a, b) satisfies $b > b^-_j(a)$, then evaluate the welfare of all candidate pairs $(\tilde{a}, \tilde{b}) \in \mathcal{A} \times \mathcal{B}$ and store the maximizing policies (a'_j, b'_j). These policies imply a value under repayment given by $V^+_{R,j+1}(a, b)$.

   (b) If the government cannot issue new debt, V^-_R:

      i. If (a, b) satisfies $(1-\delta)b \leq b^-(0)$, the continuation value is safe for any $a' \geq 0$. From Lemma 2, we know that the optimal portfolio implies:

$$V^-_{R,j+1}(a, b) = \begin{cases} V_S(a - b) & \text{if } a \geq \delta b\,. \\ u\left(y + a - \left(\frac{\delta+r}{1+r}\right)b\right) + \beta V_S(-(1-\delta)b) & \text{if } a < \delta b\,. \end{cases}$$

[44] For the baseline calibration, we used $b_{\min} = 0.89$, $b_{\max} = 1.12$, and $N_b = 922$. First, we defined the bounds for the $\mathcal{B}$ grid as follows. Let $b_S(a)$ be debt level that satisfies $V_S(a - b_S(a)) = V_D(a)$. Noticing that $b^+(a) \leq b_S(a)$, we set $b_{\max}$ slightly above $b_S(a_{\max})$. Similarly, we set $b_{\min}$ slightly below $b^-(0)$.

ii. If (a, b) satisfies $(1-\delta)b > b^{-}(0)$, then evaluate the welfare of all candidate pairs $(\tilde{a}, \tilde{b})$ and store the maximum:

$$V^{-}_{R,j+1}(a,b) = \max_{\tilde{a},\tilde{b}} \left\{ u\left(y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{\tilde{a}}{1+r} + q(\tilde{a},\tilde{b})(\tilde{b} - (1-\delta)b) \right) + \beta \mathbb{E} V_j(\tilde{a},\tilde{b}) \right\}$$

subject to $\tilde{b} \leq (1-\delta)b$, $\tilde{a} \in \mathcal{A}$ and $\tilde{b} \in \mathcal{B}$. Whenever $b' \equiv (1-\delta)b \notin \mathcal{B}$, interpolate $\mathbb{E}V_j(\tilde{a}, \cdot)$ linearly in order to consider this choice.

7. Given $V_D(a)$, $V^{+}_{R,j+1}(a,b)$ and $V^{-}_{R,j+1}(a,b)$, first compute

$$\varepsilon_v \equiv \|V_{j+1} - V_j\|_\infty, \quad \varepsilon_a \equiv \|a'_{j+1} - a'_j\|_\infty, \quad \varepsilon_b \equiv \|b'_{j+1} - b'_j\|_\infty$$

and update $V_j(a,b)$ to $V_{j+1}(a,b)$ and $\mathbb{E}V_j(a,b)$ to $\mathbb{E}V_{j+1}(a,b)$ following (8) and (11).

8. Given $V_D(a)$, $V^{+}_{R,j+1}(a,b)$, and $V^{-}_{R,j+1}(a,b)$, update $\{b^{-}_j(a), b^{+}_j(a)\}$ to $\{b^{-}_{j+1}(a), b^{+}_{j+1}(a)\}$ for each $a \in \mathcal{A}$, linearly interpolating $V^{-}_{R,j}(a,\cdot)$ and $V^{+}_{R,j}(a,\cdot)$ as needed. The three zones are now given by $\mathbf{S}_{j+1}, \mathbf{C}_{j+1}, \mathbf{D}_{j+1}$.

9. Given $\mathbf{S}_{j+1}$, $\mathbf{C}_{j+1}$, $\mathbf{D}_{j+1}$, update the bond price schedule for each $(a', b') \in \mathcal{A} \times \mathcal{B}$ as follows:

$$q_{j+1}(a',b') = \begin{cases} \frac{1}{1+r} & \text{if } (a',b') \in \mathbf{S_{j+1}}, \\ 0 & \text{if } (a',b') \in \mathbf{D_{j+1}}, \\ \frac{1-\lambda}{1+r}\left(\frac{\delta+r}{1+r} + (1-\delta)q_j(a'',b'')\right) & \text{if } (a',b') \in \mathbf{C_{j+1}}, \end{cases}$$

where (a'', b'') is the optimal portfolio under state (a', b').

10. Check for convergence. Compute $\varepsilon_q \equiv \|q_{j+1} - q_j\|_\infty$. If $\max\{\varepsilon_q, \varepsilon_v, \varepsilon_a, \varepsilon_b\} < 10^{-8}$, stop. Otherwise, set

$$q_{j+1}(a,b) = \xi q_{j+1}(a,b) + (1-\xi) q_j(a,b),$$

update the iterator counter to $j+1$, and go back to step (6). We set the damping parameter as $\xi = 0.25$.

11. After convergence, compute $(a^\star, b^\star) = \arg\min \{a - b : (a,b) \in \mathbf{S} \cap (\mathcal{A} \times \mathcal{B})\}$.

I Empirical Appendix

I.1 Database

We use the database in Sosa-Padilla and Sturzenegger (2023) as our starting point. They have data from 1997.Q1 until 2019.Q3. We extend this all the way to 2025.Q3. Here we provide a brief description of the time and country coverage, as well as the main variables and their sources.[45]

Time and country coverage. We use quarterly data from 1997.Q1 until 2025.Q3. We have 37 countries in the final panel: Algeria, Belarus, Belize, Brazil, Bulgaria, Chile, Colombia, Costa Rica, Dominican Republic, Egypt, El Salvador, Gabon, Georgia, Ghana, Hungary, Indonesia, Jamaica, Jordan, Kazakhstan, Mexico, Mongolia, Morocco, Namibia, Nigeria, Pakistan, Peru, Philippines, Romania, Russian Federation, Senegal, South Africa, Sri Lanka, Thailand, Tunisia, Turkey, Ukraine, and Venezuela.

Variables and sources. The main variables and their sources are:

1. Sovereign spreads: JP Morgan EMBI global index blended spread.
2. Reserves: Total reserves (with gold at national valuation). Taken from International Monetary Fund's International Financial Statistics.
3. Sovereign Debt: Public and publicly guaranteed debt from private creditors. Taken from the World Bank's Quarterly External Debt Statistics GDDS.
4. Private Debt: External private (non-guaranteed) debt stocks. Taken from the World Bank's Quarterly External Debt Statistics GDDS.
5. World Rate: US Treasury notes, 10-year constant maturity yield.
6. Risk aversion: Merrill Lynch ICE BofAML Option-Adjusted Spreads.

[45]We refer to the empirical appendix in Sosa-Padilla and Sturzenegger (2023) for further details.

I.2 Additional Results

Full regression table. In Table I.1 we repeat the results from Table 2 in the main text, reporting all coefficients.

Table I.1: Panel regressions (with all control variables reported)

Dep. Variable:	Spread (in bps)			
	Full Sample		Low Debt	High Debt
	(1)	(2)	(3)	(4)
Reserves	-13.51***	-10.72***	-10.28***	-18.99***
	(1.29)	(1.83)	(2.97)	(4.14)
Sov.Debt	6.85***	8.73***	10.29***	10.71***
	(0.91)	(1.27)	(2.84)	(2.68)
Reserves × Sov.Debt		-0.11**	-0.38***	0.09
		(0.05)	(0.11)	(0.10)
Private Debt	3.88***	3.72***	4.95***	2.88*
	(0.73)	(0.74)	(1.28)	(1.48)
log(Risk Aversion)	971.24***	968.28***	1798.42***	630.37***
	(117.38)	(117.27)	(184.43)	(148.04)
log(World Rate)	-203.17***	-203.53***	-135.80***	-154.57***
	(38.25)	(38.21)	(48.33)	(57.34)
Marg. Effect (in bps)	-6.65**	-5.01	-10.19*	-5.20
[p-value]	[0.043]	[0.204]	[0.084]	[0.451]
Num. Obs.	1,767	1,767	949	818
R2 Adj.	0.298	0.299	0.376	0.286

Note: All specifications include year dummies and country FEs. Reserves and Sov.Debt are measured as a percent of GDP. Robust standard errors are in parentheses. *p<0.1; **p<0.05; ***p<0.01. The "Marg. Effect" row reports the discrete marginal effect (on spreads) of increasing reserves and debt by 1% of GDP (see footnote 39).

Reserves not used during crises: Conditional evidence. Figure I.1 presents a residualized version of the relationship shown in Figure 8 in the main text. While the main-text figure plots the unconditional association between reserve changes during the crisis and reserves at the beginning of the crisis, the appendix figure removes the influence of contemporaneous crisis conditions (i.e., changes in sovereign debt, sovereign spreads, and GDP growth) from both variables before plotting.

The residualized plot reveals no statistically significant relationship between initial reserves and subsequent reserve changes once crisis conditions are accounted for: the fitted slope is close to zero and statistically insignificant (p-value = 0.76). This implies that the association visible in the raw scatter in the main text largely reflects differences in crisis intensity across countries, rather than a direct relationship between initial reserve holdings and reserve depletion.

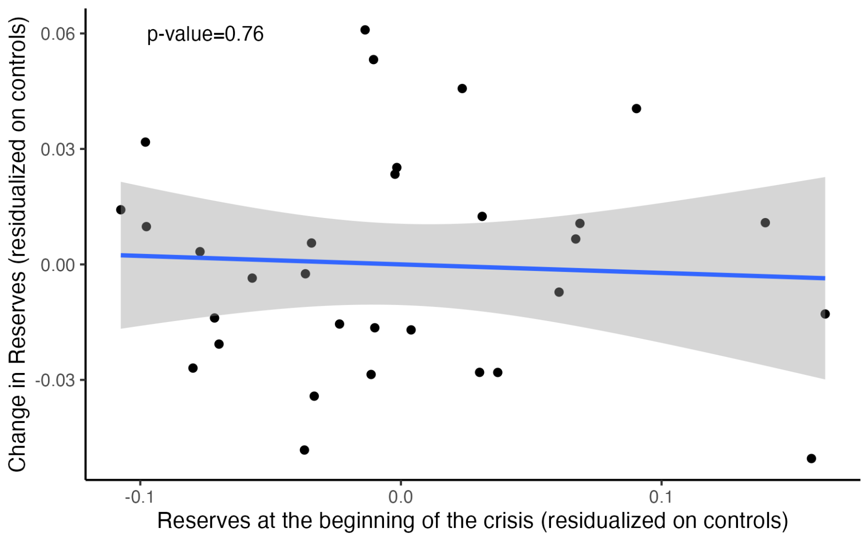

Figure I.1: Reserves not used during crises – Residualized. The figure shows the residualized relationship between initial reserves and reserve changes. Each dot represents a country-crisis observation. The figure also shows the line of best fit, together with the 95% confidence interval.